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The Total Variation Flow in $\mathbb{R}^N$

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In this paper, we study the minimizing total variation flow $u_t = \mathrm{div}(Du/|Du|)$ in $\mathbb{R}^N$ for initial data u_0 in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, proving an existence and uniqueness result. Then we characterize all bounded sets Ω of finite perimeter in $\mathbb{R}^2$ which evolve without distortion of the boundary. In that case, $u_0 = \chi_\Omega$ evolves as $u(t,x) = (1-\lambda_\Omega t)^+\chi_\Omega$, where χ_Ω is the characteristic function of Ω, $\lambda_\Omega := P(\Omega)/|\Omega|$, and $P(\Omega)$ denotes the perimeter of Ω. We give examples of such sets. The solutions are such that $v := \lambda_\Omega\chi_\Omega$ solves the eigenvalue problem $-\mathrm{div}\left(\frac{Dv}{|Dv|}\right) = v$. We construct other explicit solutions of this problem. As an application, we construct explicit solutions of the denoising problem in image processing.

Key Words: total variation flow; nonlinear parabolic equations; finite perimeter sets; calibrable sets.

1. INTRODUCTION

In this paper, we are interested in the equation

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\frac{Du}{|Du|}\right) \qquad \text{in }]0,\infty[\times\mathbb{R}^N, \tag{1}$$

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coupled with the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^N \tag{2}$$

for a given $u_0 \in L_{\mathrm{loc}}(\mathbb{R}^N)$. This PDE appears (in a bounded domain D) in the steepest descent method for minimizing the total variation, a method introduced by Rudin *et al.* [33] in the context of image denoising and reconstruction. When dealing with the deconvolution or reconstruction problem one minimizes the total variation functional

$$\int_D |Du| \tag{3}$$

with some constraints which model the process of image acquisition, including blur and noise. The constraint can be written as $z = K * u + n$, where z is the observed image, K is a convolution operator whose kernel represents the point spread function of the optical system, n is the noise and u is the ideal image, previous to distortion. The denoising problem corresponds to the case $K = I$, and the constraint becomes $z = u + n$. Then one minimizes (3) under one of the above constraints [33]. Numerical experiments show that the model is adapted to restore the discontinuities of the image [18, 25, 33, 35, 36]. Indeed, the underlying functional model is the space of BV functions, i.e., functions of bounded variation, which admit a discontinuity set which is countably rectifiable [2, 26, 38].

To solve (3) (with the specified constraint) one formally computes the Euler–Lagrange equation and solves it with Neumann boundary conditions, which amounts to a reflection of the image across the boundary of D. Many numerical methods have been proposed to solve this equation in practice, see for instance [18, 25, 33, 35, 36] (see also [31] for an interesting analysis of the features of most numerical methods explaining, in particular, the staircasing effect). This leads to an iterative process which, in some sense, can be understood as a gradient descent. Thus, to understand how total variation is minimized by functional (3) we shall forget about the constraint and study the gradient descent flow of (3). In a bounded domain, this leads to the study of (1) under Neumann boundary conditions and this study was done in [3] where the authors proved existence and uniqueness of solutions, and constructed some particular explicit solutions of the equation. This study was completed in [5] where the authors proved that the solution reaches its asymptotic state in finite time and studied its extinction profile, given in terms of the eigenvalue problem

$$-\mathrm{div}\left(\frac{Dv}{Dv|}\right) = v. \tag{4}$$

A similar study was done in [5] for Dirichlet boundary conditions. Still, we need a better understanding of the behavior of (1) when minimizing the total variation and, for that, we need to have at our disposal explicit solutions which display this behavior. To avoid technicalities due to the presence of the boundary, we will study (1) in the whole space and we will construct a family of explicit solutions corresponding to the evolution of sets, i.e., solutions whose initial condition is given by the characteristic function χ_Ω of a set Ω. In particular, in two space dimensions, we are interested in understanding for which bounded sets Ω the solution of (1) and (2) with $u_0 = \chi_\Omega$ decreases its height, without distortion of the boundary of Ω.

In this respect, a useful remark is that functional (3) can be regarded, up to a constant and on a bounded domain, as the anisotropic perimeter [12] of the set $\{(x, y) \in \mathbb{R}^N \times \mathbb{R}: y < u(x)\}$, corresponding to the anisotropy given by the cylindrical norm $\phi(z, \zeta) := \max\{\|z\|, |\zeta|\}$, for $(z, \zeta) \in \mathbb{R}^N \times \mathbb{R}$. Therefore, Eq. (1) is similar (even if not exactly the same) to the equation defining the anisotropic mean curvature flow corresponding to ϕ. Interestingly enough, it turns out that, when $N = 2$, the problem of determining those bounded connected sets Ω whose characteristic function evolve by decreasing its height is close to the problem of determining which planar horizontal facets of a given solid subset of $\mathbb{R}^2 \times \mathbb{R}$ do not break or bend under the ϕ-anisotropic mean curvature flow. This problem has been considered in [10, 11] and the techniques developed there can be adapted, to some extent, to the present situation (see in particular Theorem 4).

Let us explain the plan of the paper. In Section 2, we recall some basic facts about BV functions and the integration by parts formula. In Section 3, we study the well-posedness of (1) and (2) for initial data u_0 in $L^2(\mathbb{R}^N)$. In Section 4, we give the definition of entropy solutions of (1) and (2) for initial data u_0 in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, and we state the existence and uniqueness theorem (see Theorem 3). Sections 5 and 6 are devoted to prove the uniqueness and the existence part of Theorem 3, respectively. In Section 7, we prove the regularity in time of the entropy solution when the initial condition is bounded above or below by a constant. In Section 8, we characterize all *bounded connected* subsets Ω of $\mathbb{R}^2$ for which the solution of (1) and (2) with $u_0 = \chi_\Omega$ does not deform its boundary but only decreases its height. In Theorem 4, we prove that if $C \subset \mathbb{R}^2$ is a bounded set of finite perimeter which is connected, then the solution u of (1) and (2) with $u(0, x) = \chi_C(x)$ is given by

$$u(t, x) = (1 - \lambda_C t)^+ \chi_C(x), \qquad \lambda_C := \frac{P(C)}{|C|}$$

(where $P(C)$ stands for the perimeter of C and $|C|$ for the Lebesgue measure of C) if and only C is convex, ∂C is of class $C^{1,1}$ and

$$\operatorname*{ess\,sup}_{p \in \partial C} \kappa_{\partial C}(p) \leqslant \lambda_C, \tag{5}$$

where $\kappa_{\partial C}$ denotes the (almost everywhere defined) curvature of ∂C. The characterization for general *nonconnected* bounded sets of finite perimeter Ω is the argument of Section 9, see Theorems 6 and 7. In particular, beside the conditions of Theorem 4 on each connected component C_i of Ω, $i = 1, \dots, m$, a new property must be added in the list of necessary and sufficient conditions, which reads as follows. Let $0 \leqslant k \leqslant m$ and let $\{i_1, \dots, i_k\} \subseteq \{1, \dots, m\}$ be any k-uple of indices; if we denote by $E_{i_1,\dots,i_k}$ a solution of the variational problem

$$\min\left\{P(E)\colon \bigcup_{j=1}^{k} C_{i_j} \subseteq E \subseteq \mathbb{R}^2 \setminus \bigcup_{j=k+1}^{m} C_{i_j}\right\},$$

then

$$P(E_{i_1,\dots,i_k}) \geqslant \sum_{j=1}^{k} P(C_{i_j}). \tag{6}$$

Notice that (6) implies, in particular, a condition between the mutual distances between all sets C_i. More generally, we construct solutions of (4) of the form $\sum_{i=1}^{m} \lambda_{C_i} \chi_{C_i}$ where $\lambda_{C_i} := \frac{P(C_i)}{|C_i|}$, C_i are bounded open convex sets of class $C^{1,1}$ satisfying the curvature bound (5) and the variational property described in (6).

The previous results allow us to explicitly compute the minimum of the denoising problem

$$\min_{u \in L^2(\mathbb{R}^2) \cap \mathrm{BV}(\mathbb{R}^2)} \left\{\int_{\mathbb{R}^2} |Du| + \frac{1}{2\lambda} \int_{\mathbb{R}^2} (u - f)^2 \, dx\right\}, \tag{7}$$

where $\lambda > 0$, $f := \sum_{i=1}^{m} b_i \lambda_{C_i} \chi_{C_i}$, for $b_i \in \mathbb{R}$ and C_i sets of the type described above. Indeed, in Section 10 we prove that if the function $v := \sum_{i=1}^{m} \lambda_{C_i} \chi_{C_i}$ solves (4) then $u := \sum_{i=1}^{m} a_i \lambda_{C_i} \chi_{C_i}$ solves (7) where $a_i := \mathrm{sign}(b_i)(|b_i| - \lambda)^+$. A converse statement holds if $b_i - a_i = \lambda$, or $b_i - a_i = -\lambda$, for all $i = 1, \dots, m$. Note that a_i is given in terms of a soft thresholding of b_i with threshold λ. This is in coincidence with the soft thresholding rule applied to the wavelet coefficients of a noisy function (the uncorrupted function being in some Besov space) [22–24, 37]. Finally, in Section 11, we illustrate our results, in particular the role of condition (6), with some explicit examples.

2. SOME NOTATION

Let Q be an open subset of $\mathbb{R}^N$. A function $u \in L^1(Q)$ whose gradient Du in the sense of distributions is a (vector valued) Radon measure with finite total variation in Q is called a function of bounded variation. The class of such functions will be denoted by $\mathrm{BV}(Q)$. The total variation of Du on Q turns out to be

$$\sup\left\{\int_Q u \operatorname{div} z\, dx:\ z \in C_0^\infty(Q;\mathbb{R}^n),\ \|z\|_{L^\infty(Q)} := \operatorname{ess}\sup_{x\in Q} |z(x)| \leqslant 1\right\} \tag{8}$$

(where for a vector $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$) and will be denoted by $|Du|(Q)$ or by $\int_Q |Du|$. It turns out that the map $u \to |Du|(Q)$ is $L^1_{\mathrm{loc}}(Q)$-lower semicontinuous. $\mathrm{BV}(Q)$ is a Banach space when endowed with the norm $\int_Q |u|\, dx + |Du|(Q)$. We recall that $\mathrm{BV}(\mathbb{R}^N) \subseteq L^{N/(N-1)}(\mathbb{R}^N)$. The total variation of u on a Borel set $B \subseteq Q$ is defined as $\inf\{|Du|(A): A$ open, $B \subseteq A \subseteq Q\}$.

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter in Q if (8) is finite when u is substituted with the characteristic function χ_E of E. The perimeter of E in Q is defined as $P(E, Q) := |D\chi_E|(Q)$. We shall use the notation $P(E) := P(E, \mathbb{R}^N)$. For sets of finite perimeter E one can define the essential boundary $\partial^* E$, which is countably $(N-1)$ rectifiable with finite $\mathcal{H}^{N-1}$ measure, and compute the outer unit normal $\nu^E(x)$ at $\mathcal{H}^{N-1}$ almost all points x of $\partial^* E$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. Moreover, $|D\chi_E|$ coincides with the restriction of $\mathcal{H}^{N-1}$ to $\partial^* E$.

Each set E of finite perimeter will be identified with the representative (in its Lebesgue class) given by the set of all points $x \in \mathbb{R}^N$ such that $\lim_{\rho\to 0^+} \frac{|E\cap B_\rho(x)|}{\omega_N \rho^N} = 1$. Here $B_\rho(x)$ denotes the open ball of radius ρ centered at x, $|\cdot|$ stands for the Lebesgue measure, and ω_N is the Lebesgue measure of the unit ball of $\mathbb{R}^N$. It is clear that if ∂E is Lipschitz continuous, then the precise representative we are choosing is an open set.

We now recall [1] some basic results about connected components of sets of finite perimeter. Let $E \subseteq \mathbb{R}^N$ be a set with finite perimeter. We say that E is *decomposable* if there exists a partition (A, B) of E such that $P(E) = P(A) + P(B)$ and both $|A|$ and $|B|$ are strictly positive. We say that E is *indecomposable* if it is not decomposable; notice that the properties of being decomposable or indecomposable are invariant modulo Lebesgue null sets. It turns out that, if E is a set with finite perimeter in $\mathbb{R}^N$, there exists a unique at most countable family of pairwise disjoint (modulo $|\cdot|$) indecomposable sets $\{E_i\}_{i\in I}$ such that $|E_i| > 0$ and $P(E) = \sum_i P(E_i)$. Moreover $\mathcal{H}^{N-1}\left(E \setminus \bigcup_{i\in I} E_i\right) = 0$ and the E_i's are maximal indecomposable sets, i.e., any indecomposable set $F \subseteq E$ is contained (modulo $|\cdot|$) in some E_i. We call the sets E_i the connected components of E.

We denote by $\mathrm{BV}_{\mathrm{loc}}(Q)$ the space of functions $w \in L^1_{\mathrm{loc}}(Q)$ such that $w\varphi \in \mathrm{BV}(Q)$ for all $\varphi \in C_0^\infty(Q)$. For results and informations on functions of bounded variation we refer to [2, 26].

If μ is a (possibly vector valued) Radon measure and f is a Borel function, the integration of f with respect to μ will be denoted by $\int f d\mu$. When μ is the Lebesgue measure, the symbol dx will be often omitted.

By $L^1_w(]0,T[;\mathrm{BV}(\mathbb{R}^N))$ we denote the space of functions $w:[0,T]\to \mathrm{BV}(\mathbb{R}^N)$ such that $w \in L^1(]0,T[\times\mathbb{R}^N)$, the maps $t \in [0,T] \to \int_{\mathbb{R}^N} \phi \, dDw(t)$ are measurable for every $\phi \in C^1_0(\mathbb{R}^N;\mathbb{R}^N)$ and $\int_0^T |Dw(t)|(\mathbb{R}^N)\,dt < \infty$. By $L^1_w(]0,T[;\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N))$ we denote the space of functions $w:[0,T]\to \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N)$ such that $w\varphi \in L^1_w(]0,T[;\mathrm{BV}(\mathbb{R}^N))$ for all $\varphi \in C^\infty_0(\mathbb{R}^N)$.

Following [8], let

$$X_2(\mathbb{R}^N) := \{z \in L^\infty(\mathbb{R}^N;\mathbb{R}^N):\ \operatorname{div} z \in L^2(\mathbb{R}^N)\}.$$

If $z \in X_2(\mathbb{R}^N)$ and $w \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N)$ we define the functional $(z,Dw): C^\infty_0(\mathbb{R}^N) \to \mathbb{R}$ by the formula

$$\langle (z,Dw),\varphi\rangle := -\int_{\mathbb{R}^N} w\varphi \operatorname{div} z\,dx - \int_{\mathbb{R}^N} w\, z\cdot\nabla\varphi\,dx.$$

Then (z,Dw) is a Radon measure in $\mathbb{R}^N$,

$$\int_{\mathbb{R}^N} (z,Dw) = \int_{\mathbb{R}^N} z\cdot\nabla w\,dx \qquad \forall w \in L^2(\mathbb{R}^N)\cap W^{1,1}(\mathbb{R}^N)$$

and

$$\left|\int_B (z,Dw)\right| \leqslant \int_B |(z,Dw)| \leqslant \|z\|_\infty \int_B |Dw| \qquad \forall B \text{ Borel set } \subseteq \mathbb{R}^N. \quad (9)$$

Moreover, we have the following integration by parts formula [8], for $z \in X_2(\mathbb{R}^N)$ and $w \in L^2(\mathbb{R}^N)\cap \mathrm{BV}(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} w \operatorname{div} z\,dx + \int_{\mathbb{R}^N} (z,Dw) = 0. \quad (10)$$

We denote by $\theta(z,Dw) \in L^\infty_{|Dw|}(\mathbb{R}^N)$ the density of (z,Dw) with respect to $|Dw|$, that is

$$(z,Dw)(B) = \int_B \theta(z,Dw)\,d|Dw| \qquad \text{for any Borel set } B \subseteq \mathbb{R}^N. \quad (11)$$

In particular, if Ω is bounded and has finite perimeter in $\mathbb{R}^N$, from (10) and (11) it follows that

$$\int_\Omega \operatorname{div} z\,dx = \int_{\mathbb{R}^N} (z,-D\chi_\Omega) = \int_{\partial^*\Omega} \theta(z,-D\chi_\Omega)\,d\mathscr{H}^{N-1}. \quad (12)$$

Notice also that if $z_1, z_2 \in X_2(\mathbb{R}^N)$ and $z_1 = z_2$ almost everywhere on Ω, then $\theta(z_1, -D\chi_\Omega)(x) = \theta(z_2, -D\chi_\Omega)(x)$ for $\mathscr{H}^{N-1}$-almost every $x \in \partial^*\Omega$.

We recall the following result proved in [8].

THEOREM 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $u \in \mathrm{BV}(\Omega)$ and $z \in L^\infty(\Omega; \mathbb{R}^N)$ with $\operatorname{div} z \in L^N(\Omega)$. Then there exists a function $[z \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ such that $\|[z \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \leqslant \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and*

$$\int_\Omega u \operatorname{div} z \, dx + \int_\Omega \theta(z, Du)\, d|Du| = \int_{\partial\Omega} [z \cdot \nu^\Omega] u \, d\mathscr{H}^{N-1}. \tag{13}$$

In particular, if Ω is a bounded open set with Lipschitz boundary, then (12) has a meaning also if z is defined only on Ω and not on the whole of $\mathbb{R}^N$, precisely when $z \in L^\infty(\Omega; \mathbb{R}^N)$ with $\operatorname{div} z \in L^N(\Omega)$. In this case we mean that $\theta(z, -D\chi_\Omega)$ coincides with $[z \cdot \nu^\Omega]$.

Remark 1. Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz open set, and let $z_{\mathrm{inn}} \in L^\infty(\Omega; \mathbb{R}^2)$ with $\operatorname{div} z_{\mathrm{inn}} \in L^2_{\mathrm{loc}}(\Omega)$, and $z_{\mathrm{out}} \in L^\infty(\mathbb{R}^2 \backslash \bar{\Omega}; \mathbb{R}^2)$ with $\operatorname{div} z_{\mathrm{out}} \in L^2_{\mathrm{loc}}(\mathbb{R}^2 \backslash \bar{\Omega})$. Assume that

$$\theta(z_{\mathrm{inn}}, -D\chi_\Omega)(x) = -\theta(z_{\mathrm{out}}, -D\chi_{\mathbb{R}^2 \backslash \bar{\Omega}})(x) \qquad \text{for } \mathscr{H}^1 - \text{a.e } x \in \partial\Omega.$$

Then if we define $z := z_{\mathrm{inn}}$ on Ω and $z := z_{\mathrm{out}}$ on $\mathbb{R}^2 \backslash \bar{\Omega}$, we have $z \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ and $\operatorname{div} z \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$.

3. INITIAL CONDITIONS IN $L^2(\mathbb{R}^N)$

Throughout the paper, given a (possibly vector valued) function f depending on space and time, we usually write $f(t)$ to mean the function $f(t, \cdot)$.

DEFINITION 1. A function $u \in C([0, T]; L^2(\mathbb{R}^N))$ is called a strong solution of (1) if

$$u \in W^{1,2}_{\mathrm{loc}}(0, T; L^2(\mathbb{R}^N)) \cap L^1_w(]0, T[; \mathrm{BV}(\mathbb{R}^N))$$

and there exists $z \in L^\infty(]0, T[\times\mathbb{R}^N; \mathbb{R}^N)$ with $\|z\|_\infty \leqslant 1$ such that

$$u_t = \operatorname{div} z \qquad \text{in } \mathscr{D}'(]0, T[\times\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} (u(t) - w)u_t(t) = \int_{\mathbb{R}^N} (z(t), Dw) - \int_{\mathbb{R}^N} |Du(t)|$$
$$\forall w \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N), \quad \text{a.e. } t \in [0, T]. \tag{14}$$

The aim of this section is to prove the following result.

THEOREM 2. *Let $u_0 \in L^2(\mathbb{R}^N)$. Then there exists a unique strong solution u of* (1), (2) *in* $[0, T] \times \mathbb{R}^N$ *for every* $T > 0$. *Moreover, if u and v are the strong solutions of* (1) *corresponding to the initial conditions* $u_0, v_0 \in L^2(\mathbb{R}^N)$, *then*

$$\|(u(t) - v(t))^+\|_2 \leqslant \|(u_0 - v_0)^+\|_2 \qquad \text{for any } t > 0. \tag{15}$$

Proof. Let us introduce the following multivalued operator $\mathscr{A}$ in $L^2(\mathbb{R}^N)$: a pair of functions (u, v) belongs to the graph of $\mathscr{A}$ if and only if

$$u \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N), \qquad v \in L^2(\mathbb{R}^N), \tag{16}$$

$$\text{there exists } z \in X_2(\mathbb{R}^N) \text{ with } \|z\|_\infty \leqslant 1, \text{ such that } v = -\operatorname{div} z \tag{17}$$

and

$$\int_{\mathbb{R}^N} (w - u)v \leqslant \int_{\mathbb{R}^N} z \cdot \nabla w - \int_{\mathbb{R}^N} |Du| \qquad \forall w \in L^2(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N).$$

Let also $\Psi : L^2(\mathbb{R}^N) \to]-\infty, +\infty]$ be the functional defined by

$$\Psi(u) := \begin{cases} \int_{\mathbb{R}^N} |Du| & \text{if } u \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N), \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus \mathrm{BV}(\mathbb{R}^N). \end{cases} \tag{18}$$

Since Ψ is convex and lower semicontinuous in $L^2(\mathbb{R}^N)$, its subdifferential $\partial\Psi$ is a maximal monotone operator in $L^2(\mathbb{R}^N)$.

We divide the proof of the theorem into three steps.

Step 1. The following assertions are equivalent:

(a) $(u, v) \in \mathscr{A}$;

(b) (16) and (17) hold,

and

$$\int_{\mathbb{R}^N} (w - u)v \leqslant \int_{\mathbb{R}^N} (z, Dw) - \int_{\mathbb{R}^N} |Du| \qquad \forall w \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N); \tag{19}$$

(c) (16) and (17) hold, and (19) holds with the equality instead of the inequality;

(d) (16) and (17) hold, and

$$\int_{\mathbb{R}^N} (z, Du) = \int_{\mathbb{R}^N} |Du|. \tag{20}$$

It is clear that (c) implies (b), and (b) implies (a), while (d) follows from (b) with the choice $w = u$ using (9). In order to prove that (a) implies (b) it is enough to use Lemmas 5.2 and 1.8 of [8]. To obtain (c) from (d) it suffices to multiply both terms of the equation $v = -\operatorname{div} z$ by $w - u$, for $w \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N)$ and to integrate by parts using (10).

Step 2. The operator $\mathscr{A}$ is maximal monotone in $L^2(\mathbb{R}^N)$ with dense domain. The proof of the monotonicity of $\mathscr{A}$ follows from (c) of Step 1 and (10). Note also that, as a consequence of Step 1, one can prove that $\mathscr{A}$ is closed. The other assertions can be proved as in [3, 4]. Indeed, if $f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ has compact support, using the idea of approximating $\mathscr{A}$ with the p-Laplace operator (see [3, 4]), one can prove that, if $\lambda > 0$, there exists a solution u of

$$u + \lambda \mathscr{A} u = f. \tag{21}$$

The closedness of $\mathscr{A}$ implies that (21) can be solved for any $f \in L^2(\mathbb{R}^N)$. It follows that the range of $I + \lambda\mathscr{A}$ is the whole of $L^2(\mathbb{R}^N)$, and therefore $\mathscr{A}$ is maximal monotone. The density of the domain of $\mathscr{A}$ can be proved as in [3].

Step 3. We also have $\mathscr{A} = \partial\Psi$. The proof is similar to the proof of Lemma 1 in [4] and we omit the details.

As a consequence, the semigroup generated by $\mathscr{A}$ coincides with the semigroup generated by $\partial\Psi$ and therefore (see [16]) $u(t,x) = e^{-t\mathscr{A}} u_0(x)$ is a strong solution of

$$u_t + \mathscr{A} u \ni 0,$$

i.e., $u \in W^{1,2}_{\mathrm{loc}}(]0, T[; L^2(\mathbb{R}^N))$ and $-u_t(t) \in \mathscr{A} u(t)$ for almost all $t \in\,]0, T[$ [16, Theorem 3.1]. Then, according to the equivalence proved in Step 1, we have that

$$\begin{aligned} &\int_{\mathbb{R}^N} (u(t) - w) u_t(t) = \int_{\mathbb{R}^N} (z(t), Dw) - \int_{\mathbb{R}^N} |Du(t)| \\ &\forall w \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N) \end{aligned} \tag{22}$$

for almost all $t \in]0, T[$. Now, choosing $w = u - \varphi$, $\varphi \in C_0^\infty(\mathbb{R}^N)$, we see that $u_t(t) = \operatorname{div} z(t)$ in $\mathscr{D}'(\mathbb{R}^N)$ for almost every $t \in]0, T[$. We deduce that $u_t = \operatorname{div} z$

in $\mathscr{D}'(]0,T[\times\mathbb{R}^N)$. We have proved that u is a strong solution of (1) in the sense of Definition 1.

The contractivity estimate (15) of Theorem 2 follows as in [3, 4]. This concludes the proof of the Theorem. ∎

Given a function $g \in L^2(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$, we define

$$\|g\|_* := \sup\left\{\left|\int_{\mathbb{R}^N} g(x)u(x)\,dx\right| : u \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |Du| \leqslant 1\right\}.$$

Part (b) of the next lemma gives a characterization of $\mathscr{A}0$ which will be useful in Section 9 to find vector fields whose divergence is assigned. This part of the lemma was proved in [30] in the context of the analysis of the Rudin–Osher–Fatemi model for image denoising; for the sake of completeness, we shall include its proof.

Lemma 1. *Let $f \in L^2(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$ and $\lambda > 0$. The following assertions hold*:

(a) *The function u is the solution of*

$$\min_{w \in L^2(\mathbb{R}^N)\cap \mathrm{BV}(\mathbb{R}^N)} D(w), \qquad D(w) := \int_{\mathbb{R}^N} |Dw| + \frac{1}{2\lambda}\int_{\mathbb{R}^N} (w-f)^2\,dx \quad (23)$$

if and only if there exists $z \in X_2(\mathbb{R}^N)$ satisfying (20) *with $\|z\|_\infty \leqslant 1$ and $-\lambda\,\mathrm{div}\, z = f - u$.*

(b) *The function $u \equiv 0$ is the solution of* (23) *if and only if $\|f\|_* \leqslant \lambda$.*

(c) If $N = 2$, $\mathscr{A}0 = \{f \in L^2(\mathbb{R}^2) : \|f\|_* \leqslant 1\}$.

Proof. (a) Thanks to the strict convexity of D, u is the solution of (23) if and only if $0 \in \partial D(u) = \partial\Psi(u) + (u - f) = \mathscr{A}(u) + (u - f)$, where Ψ is defined in (18) and the last equality follows from Step 3 in the proof of Theorem 2. This is equivalent to $-\lambda\,\mathrm{div}(\frac{Du}{|Du|}) = f - u$, i.e., there exists $z \in X_2(\mathbb{R}^N)$ satisfying (20) with $\|z\|_\infty \leqslant 1$ and $-\lambda\,\mathrm{div}\, z = f - u$ (recall the definition of $\mathscr{A}$ in the proof of Theorem 2).

(b) The function $u \equiv 0$ is the solution of (23) if and only if

$$\int_{\mathbb{R}^N} |Dv| + \frac{1}{2\lambda}\int_{\mathbb{R}^N} (v-f)^2\,dx \geqslant \frac{1}{2\lambda}\int_{\mathbb{R}^N} f^2\,dx \qquad \forall v \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N). \quad (24)$$

Replacing v by εv (where $\varepsilon > 0$), expanding the L^2-norm, dividing by $\varepsilon > 0$, and letting $\varepsilon \to 0+$ we have

$$\left| \int_{\mathbb{R}^N} f(x)v(x)\,dx \right| \leqslant \lambda \int_{\mathbb{R}^N} |Dv| \qquad \forall v \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N). \tag{25}$$

Since (25) implies (24), we have that (24) and (25) are equivalent. The assertion follows by observing that (25) is equivalent to $\|f\|_* \leqslant \lambda$.

(c) Let $N = 2$. We have $\mathscr{A}0 = \{v \in L^2(\mathbb{R}^2) : \exists z \in X_2(\mathbb{R}^2), \|z\|_\infty \leqslant 1, -\operatorname{div} z = v\}$. On the other hand, from (a) and (b) it follows that $\|f\|_* \leqslant 1$ if and only if there exists $z \in X_2(\mathbb{R}^2)$ with $\|z\|_\infty \leqslant 1$ and such that $f = -\operatorname{div} z$. Then the assertion follows. ∎

Let us give a heuristic explanation of what the vector field z represents. Condition (20) essentially means that z has unit norm and is orthogonal to the level sets of u. In some sense, z is invariant under local contrast changes. To be more precise, we observe that if $u = \sum_{i=1}^p c_i \chi_{B_i}$ where B_i are sets of finite perimeter such that $\mathscr{H}^{N-1}((B_i \cup \partial^* B_i) \cap (B_j \cup \partial^* B_j)) = 0$ for $i \neq j$, $c_i \in \mathbb{R}$, and

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) = f \in L^2(\mathbb{R}^N), \tag{26}$$

then also $-\operatorname{div}\left(\frac{Dv}{|Dv|}\right) = f$ for any $v = \sum_{i=1}^p d_i \chi_{B_i}$ where $d_i \in \mathbb{R}$ and $\operatorname{sign}(d_i) = \operatorname{sign}(c_i)$. Indeed, there is a vector field $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ such that $\|z\|_\infty \leqslant 1$, $-\operatorname{div} z = f$ and (20) holds. Then one can check that $|D\chi_{B_i}| = \operatorname{sign}(c_i)(z, D\chi_{B_i})$ as measures in $\mathbb{R}^N$ and, as a consequence $(z, Dv) = |Dv|$ as measures in $\mathbb{R}^N$.

Let us also observe that the solutions of (26) are not unique. Indeed, if $u \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N)$ is a solution of (26) and $g \in C^1(\mathbb{R})$ with $g'(r) > 0$ for all $r \in \mathbb{R}$, then $w = g(u)$ is also a solution of (26). In other words, a global contrast change of u produces a new solution of (26). In an informal way, the previous remark can be rephrased by saying that also local contrast changes of a given solution of (26) produce new solutions of it. To express this nonuniqueness in a more general way we suppose that $(u_1, v), (u_2, v) \in \mathscr{A}$, i.e., there are vector fields $z_i \in X_2(\mathbb{R}^N)$ with $\|z_i\|_\infty \leqslant 1$, such that

$$-\operatorname{div} z_i = v, \qquad \int_{\mathbb{R}^N} (z_i, Du_i) = \int_{\mathbb{R}^N} |Du_i|, \qquad i = 1, 2.$$

Then

$$0 = -\int_{R^N} (\operatorname{div} z_1 - \operatorname{div} z_2)(u_1 - u_2)\, dx = \int_{R^N} (z_1 - z_2, Du_1 - Du_2)$$
$$= \int_{R^N} |Du_1| - (z_2, Du_1) + \int_{\mathbb{R}^N} |Du_2| - (z_1, Du_2).$$

Hence

$$\int_{R^N} |Du_1| = \int_{\mathbb{R}^N} (z_2, Du_1) \quad \text{and} \quad \int_{\mathbb{R}^N} |Du_2| = \int_{\mathbb{R}^N} (z_1, Du_2).$$

In other words, z_1 is in some sense a unit vector field of normals to the level sets of u_2 and a similar thing can be said of z_2 with respect to u_1. Any two solutions of (26) should be related in this way.

The following estimate, which is a consequence of the homogeneity of $\mathscr{A}$ [14] will be useful to prove the regularity in time of the solution when the initial condition is in $L^1_{\text{loc}}(\mathbb{R}^N)$ (see Lemma 4 of Section 7).

Proposition 1. *Let* $u_0 \in L^2(\mathbb{R}^N)$, $u_0 \geqslant 0$, *and let* u *be the strong solution of* (1) *and* (2). *Then*

$$u'(t) \leqslant \frac{u(t)}{t} \qquad \textit{for a.e. } t > 0.$$

Moreover, if $u_0 \leqslant 0$, *then* $u'(t) \geqslant \frac{u(t)}{t}$ *for almost every* $t > 0$.

Proof. We consider the case $u_0 \geqslant 0$, the other case being similar. First, let us prove that for any $\lambda > 0$, and any $t > 0$, we have that

$$\lambda^{-1} u(\lambda t) = e^{-t\mathscr{A}}(\lambda^{-1} u_0). \tag{27}$$

By Crandall–Liggett's exponential formula $e^{-t\mathscr{A}}(u_0) = \lim_{n\to\infty}(I + \frac{t}{n}\mathscr{A})^{-n} \times (u_0)$ in $L^2(\mathbb{R}^N)$ [21], it is enough to prove that for all $\mu > 0$,

$$(I + \mu\mathscr{A})^{-1}(\lambda^{-1} u_0) = \lambda^{-1}(I + \lambda\mu\mathscr{A})^{-1}(u_0). \tag{28}$$

We have $v_\mu = (I + \mu\mathscr{A})^{-1}(\lambda^{-1} u_0)$ if and only if $(v_\mu, \frac{\lambda^{-1}u_0 - v_\mu}{\mu}) \in \mathscr{A}$, which is equivalent to the existence of $z_\mu \in X_2(\mathbb{R}^N)$ such that

$$-\operatorname{div} z_\mu = \frac{\lambda^{-1} u_0 - v_\mu}{\mu},$$

$$\int_{\mathbb{R}^N} (z_\mu, Dv_\mu) = \int_{\mathbb{R}^N} |Dv_\mu|.$$

Then, we have

$$-\operatorname{div} z_\mu = \frac{u_0 - \lambda v_\mu}{\lambda\mu},$$

$$\int_\Omega (z_\mu, D(\lambda v_\mu)) = \int_{\mathbb{R}^N} |D(\lambda v_\mu)|,$$

which is equivalent to say that $\left(\lambda v_\mu, \frac{u_0 - \lambda v_\mu}{\lambda\mu}\right) \in \mathscr{A}$, that is, $v_\mu = \lambda^{-1}(I + \lambda\mu\mathscr{A})^{-1}(u_0)$, and (28) holds.

Fix $t > 0$ a differentiability point of u. For $h > 0$, let λ be such that $\lambda t = t + h$. Now, applying (27), we obtain

$$\begin{aligned} u(t+h) - u(t) &= u(\lambda t) - u(t) = (1 - \lambda^{-1})u(\lambda t) + \lambda^{-1}u(\lambda t) - u(t) \\ &= \frac{h}{t+h}\, u(t+h) + e^{-t\mathscr{A}}(\lambda^{-1}u_0) - u(t). \end{aligned}$$

Now, since $\lambda^{-1}u_0 \leqslant u_0$, by Theorem 2 we get $e^{-t\mathscr{A}}(\lambda^{-1}u_0) \leqslant u(t)$. Hence

$$u(t+h) - u(t) \leqslant \frac{h}{t+h}\, u(t+h),$$

and the result follows. ∎

4. THE NOTION OF ENTROPY SOLUTION

Let

$$\mathscr{P} := \{ p \in W^{1,\infty}(\mathbb{R}) \colon p' \geqslant 0,\ \operatorname{supp}(p') \text{ compact} \}.$$

Definition 2. A function $u \in C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ is called an entropy solution of (1), (2) if $u(t)$ converges to u_0 in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \to 0^+$,

$$p(u) \in L^1_w(]0, T[; \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N)) \qquad \forall p \in \mathscr{P},$$

and there exists $z \in L^\infty(]0, T[\times\mathbb{R}^N; \mathbb{R}^N)$ with $\|z\|_\infty \leqslant 1$ such that

$$u_t = \operatorname{div} z \qquad \text{in } \mathscr{D}'(]0, T[\times\mathbb{R}^N) \tag{29}$$

and

$$\begin{aligned} &- \int_0^T \int_{\mathbb{R}^N} j(u - l)\eta_t + \int_0^T \int_{\mathbb{R}^N} \eta\, d|D(p(u - l))| \\ &\quad + \int_0^T \int_{\mathbb{R}^N} z \cdot \nabla\eta\, p(u - l) \leqslant 0 \end{aligned} \tag{30}$$

for all $l \in \mathbb{R}$, all $\eta \in C^\infty(]0,T[\times\mathbb{R}^N)$, with $\eta \geqslant 0$, $\eta(t,x) = \phi(t)\psi(x)$, being $\phi \in C_0^\infty(]0,T[)$, $\psi \in C_0^\infty(\mathbb{R}^N)$, and all $p \in \mathscr{P}$, where $j(r) := \int_0^r p(s)\,ds$.

The notion of entropy solution for scalar conservation laws was introduced by Kruzhkov [29] in order to prove their uniqueness and the L^1 contractivity estimate using the doubling variables technique. Carrillo [17] was the first to apply Kruzhkov's method to parabolic equations, and more recently, Benilan *et al.* [13] introduced the notion of entropy solution for elliptic equations in divergence form in order to prove uniqueness when the right-hand side is a function in L^1. The case of parabolic equations was considered by Andreu *et al.* [7]. In all these cases, the elliptic operator was in divergence form and it excluded the case of operators derived from functionals with linear growth in Du. The case of the total variation with Neumann and Dirichlet boundary conditions was considered in [3, 4], respectively, and the general case was considered in [6].

Inequality (30) is a weak way to impose equality (14); indeed if we integrate by parts, we formally substitute (29), using $\|z\|_\infty \leqslant 1$ and the fact that η is nonnegative, we get

$$\begin{aligned}\int_{\mathbb{R}^N} z\cdot\nabla\eta\, p(u-l) &= -\int_{\mathbb{R}^N} j(u-l)_t\eta - \int_{\mathbb{R}^N} \eta\, d(z, D(p(u-l)))\\ &\geqslant -\int_{\mathbb{R}^N} j(u-l)_t\,\eta - \int_{\mathbb{R}^N} \eta\, d|(z, D(p(u-l))|,\end{aligned}$$

which, after integration in time, shows that the opposite inequality in (30) is satisfied.

Remark 2. If $u_0 \in L^2(\mathbb{R}^N)$, then the strong solution of (1) and (2) coincides with the entropy solution, see Lemma 2 in Section 6.

The aim of Sections 5 and 6 is to prove the following result.

THEOREM 3. *Let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then there exists a unique entropy solution of* (1) *and* (2) *in* $[0,T]\times\mathbb{R}^N$ *for all* $T>0$. *Moreover, if $u_0, u_{0k} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are such that $u_{0k}\to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and u, u_k denote the corresponding entropy solutions, then $u_k \to u$ in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$ as $k\to+\infty$.*

5. UNIQUENESS IN $L^1_{\mathrm{loc}}(\mathbb{R}^N)$

Let $\alpha > N$, $T_k(r) := \max(\min(r,k),-k)$, $T_k^+(r) = \max(T_k(r),0)$ $(k\geqslant 0)$ and let j_α be the primitive of $\alpha T_k^+(r)^{\alpha-1}$ vanishing at $r=0$. If $N=1$, we take $\alpha\geqslant 2$, so that $j'_\alpha \in W^{1,\infty}(\mathbb{R})$.

PROPOSITION 2. *Let $u_0, \bar{u}_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Let $u, \bar{u}$ be two entropy solutions of* (1) *with initial conditions $u_0, \bar{u}_0$, respectively. Then*

$$\int_{\mathbb{R}^N} j_\alpha(u(t) - \bar{u}(t)) \leqslant \int_{\mathbb{R}^N} j_\alpha(u_0 - \bar{u}_0) \qquad \forall t > 0. \tag{31}$$

Proof. Let $T > 0$ and $Q_T :=]0, T[\times\mathbb{R}^N$. Write $j = j_\alpha$, $j^*(r) := j(-r)$, $p(r) := \alpha T_k^+(r)^{\alpha-1}$, $p^*(r) := j^{*'}(r) = -p(-r)$. Let $z, \bar{z} \in L^\infty(Q_T; \mathbb{R}^N)$ with $\|z\|_\infty \leqslant 1$, $\|\bar{z}\|_\infty \leqslant 1$ and such that, if $r, \bar{r} \in \mathbb{R}^N$, with $\|r\| \leqslant 1$, $\|\bar{r}\| \leqslant 1$ and $l_1, l_2 \in \mathbb{R}$, then

$$\begin{aligned} &- \int_0^T \int_{\mathbb{R}^N} j(u - l_1)\eta_t + \int_0^T \int_{\mathbb{R}^N} \eta \, d|D(p(u - l_1))| \\ &+ \int_0^T \int_{\mathbb{R}^N} (z - r) \cdot \nabla\eta \;\; p(u - l_1) + \int_0^T \int_{\mathbb{R}^N} r \cdot \nabla\eta \, p(u - l_1) \leqslant 0, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &- \int_0^T \int_{\mathbb{R}^N} j^*(\bar{u} - l_2)\eta_t + \int_0^T \int_{\mathbb{R}^N} \eta \, d|D(p^*(\bar{u} - l_2))| \\ &+ \int_0^T \int_{\mathbb{R}^N} (\bar{z} - \bar{r}) \cdot \nabla\eta \, p^*(\bar{u} - l_2) + \int_0^T \int_{\mathbb{R}^N} \bar{r} \cdot \nabla\eta \, p^*(\bar{u} - l_2) \leqslant 0, \end{aligned} \tag{33}$$

for all $\eta \in C^\infty(Q_T)$, with $\eta \geqslant 0$, $\eta(t,x) = \phi(t)\psi(x)$, being $\phi \in C_0^\infty(]0, T[)$, $\psi \in C_0^\infty(\mathbb{R}^N)$.

We choose two different pairs of variables (t, x), (s, y) and consider u, z as functions of (t, x) and $\bar{u}$, $\bar{z}$ as functions of (s, y). Let $0 \leqslant \phi \in C_0^\infty(]0, T[)$, $0 \leqslant \psi \in C_0^\infty(\mathbb{R}^N)$, (ρ_n) a standard sequence of mollifiers in $\mathbb{R}^N$ and $(\tilde{\rho}_n)$ a sequence of mollifiers in $\mathbb{R}$. Define

$$\eta_n(t, x, s, y) := \tilde{\rho}_n(t - s)\rho_n(x - y)\phi\left(\frac{t+s}{2}\right)\psi\left(\frac{x+y}{2}\right) \geqslant 0.$$

Note that for n sufficiently large,

$$(t, x) \mapsto \eta_n(t, x, s, y) \in C_0^\infty(]0, T[\times\mathbb{R}^N) \qquad \forall (s, y) \in Q_T,$$

$$(s, y) \mapsto \eta_n(t, x, s, y) \in C_0^\infty(]0, T[\times\mathbb{R}^N) \qquad \forall (t, x) \in Q_T.$$

Hence, for (s, y) fixed, if we take $l_1 = \bar{u}(s, y)$ and $r = \bar{z}(s, y)$ in (32), we get

$$
\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^N} j(u - \bar{u}(s, y))(\eta_n)_t + \int_0^T \int_{\mathbb{R}^N} \eta_n \, d|D_x(p(u - \bar{u}(s, y)))| \\
&\quad + \int_0^T \int_{\mathbb{R}^N} (z - \bar{z}(s, y)) \cdot \nabla_x \eta_n \, p(u - \bar{u}(s, y)) \\
&\quad + \int_0^T \int_{\mathbb{R}^N} \bar{z}(s, y) \cdot \nabla_x \eta_n \, p(u - \bar{u}(s, y)) \leqslant 0. \qquad (34)
\end{aligned}
$$

Similarly, for (t, x) fixed, if we take $l_2 = u(t, x)$ and $\bar{r} = z(t, x)$ in (33), we get

$$
\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^N} j^*(\bar{u} - u(t, x))(\eta_n)_s + \int_0^T \int_{\mathbb{R}^N} \eta_n \, d|D_y(p^*(\bar{u} - u(t, x)))| \\
&\quad + \int_0^T \int_{\mathbb{R}^N} (\bar{z} - z(t, x)) \cdot \nabla_y \eta_n \, p^*(\bar{u} - u(t, x)) \\
&\quad + \int_0^T \int_{\mathbb{R}^N} z(t, x) \cdot \nabla_y \eta_n \;\; p^*(\bar{u} - u(t, x)) \leqslant 0. \qquad (35)
\end{aligned}
$$

Now, since $p^*(r) = -p(-r)$ and $j^*(r) = j(-r)$, we can rewrite (35) as

$$
\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^N} j(u(t, x) - \bar{u})(\eta_n)_s + \int_0^T \int_{\mathbb{R}^N} \eta_n \, d|D_y(p(u(t, x) - \bar{u}))| \\
&\quad + \int_0^T \int_{\mathbb{R}^N} (z(t, x) - \bar{z}) \cdot \nabla_y \eta_n \, p(u(t, x) - \bar{u}) \\
&\quad - \int_0^T \int_{\mathbb{R}^N} z(t, x) \cdot \nabla_y \eta_n \, p(u(t, x) - \bar{u}) \leqslant 0. \qquad (36)
\end{aligned}
$$

Integrating (34) with respect to (s, y) and (36) with respect to (t, x) and taking the sum yields

$$
\begin{aligned}
&- \int_{Q_T \times Q_T} j(u(t, x) - \bar{u}(s, y))((\eta_n)_t + (\eta_n)_s) \\
&\quad + \int_{Q_T \times Q_T} \eta_n \, d|D_x(p(u - \bar{u}(s, y)))| + \int_{Q_T \times Q_T} \eta_n \, d|D_y(p(u(t, x) - \bar{u}(s))| \\
&\quad + \int_{Q_T \times Q_T} (z(t, x) - \bar{z}(s, y)) \cdot (\nabla_x \eta_n + \nabla_y \eta_n) p(u(t, x) - \bar{u}(s, y)) \\
&\quad + \int_{Q_T \times Q_T} \bar{z}(s, y) \cdot \nabla_x \eta_n \;\, p(u(t, x) - \bar{u}(s, y)) \\
&\quad - \int_{Q_T \times Q_T} z(t, x) \cdot \nabla_y \eta_n \;\, p(u(t, x) - \bar{u}(s, y)) \leqslant 0. \qquad (37)
\end{aligned}
$$

Now, by Green's formula we have

$$\int_{Q_T\times Q_T} \bar{z}(s,y)\cdot\nabla_x\eta_n\, p(u(t,x)-\bar{u}(s,y)) + \int_{Q_T\times Q_T} \eta_n\, d|D_x(p(u(t,x)-\bar{u}(s,y)))|$$
$$= -\int_{Q_T\times Q_T} \eta_n(\bar{z}(s,y), D_x p(u(t,x)-\bar{u}(s,y)))$$
$$+ \int_{Q_T\times Q_T} \eta_n\, d|D_x(p(u(t,x)-\bar{u}(s,y)))| \geqslant 0$$

and

$$-\int_{Q_T\times Q_T} z(t,x)\cdot\nabla_y\eta_n\, p(u(t,x)-\bar{u}(s,y))$$
$$+ \int_{Q_T\times Q_T} \eta_n\, d|D_y(p(u(t,x)-\bar{u}(s,y)))|$$
$$= \int_{Q_T\times Q_T} \eta_n\,(z(t,x), D_y p(u(t,x)-\bar{u}(s,y)))$$
$$+ \int_{Q_T\times Q_T} \eta_n\, d|D_y(p(u(t,x)-\bar{u}(s,y)))| \geqslant 0.$$

Hence, from (37), it follows that

$$-\int_{Q_T\times Q_T} j(u(t,x)-\bar{u}(s,y))((\eta_n)_t+(\eta_n)_s)$$
$$+\int_{Q_T\times Q_T} (z(t,x)-\bar{z}(s,y))\cdot(\nabla_x\eta_n+\nabla_y\eta_n)\, p(u(t,x)-\bar{u}(s,y)) \leqslant 0. \tag{38}$$

Since

$$(\eta_n)_t+(\eta_n)_s = \tilde{\rho}_n(t-s)\rho_n(x-y)\phi'\Big(\frac{t+s}{2}\Big)\psi\Big(\frac{x+y}{2}\Big)$$

and

$$\nabla_x\eta_n+\nabla_y\eta_n = \tilde{\rho}_n(t-s)\rho_n(x-y)\phi\Big(\frac{t+s}{2}\Big)\nabla\psi\Big(\frac{x+y}{2}\Big),$$

passing to the limit in (38) as $n\to+\infty$ yields

$$-\int_{Q_T} j(u(t,x)-\bar{u}(t,x))\phi'(t)\psi(x)$$
$$+\int_{Q_T} (z(t,x)-\bar{z}(t,x))\cdot\nabla\psi(x)\;\;\phi(t)\, p(u(t,x)-\bar{u}(t,x)) \leqslant 0. \tag{39}$$

Let us choose $\psi = \varphi^\alpha$, $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geqslant 0$. Since (39) holds for any $\phi \in C_0^\infty(]0, T[)$, it follows that

$$\frac{d}{dt} \int_{\mathbb{R}^N} j(u(t,x) - \bar{u}(t,x))\varphi(x)^\alpha \\ \leqslant \int_{\mathbb{R}^N} (\bar{z}(t,x) - z(t,x)) \cdot \nabla \varphi(x)^\alpha \, p(u(t,x) - \bar{u}(t,x)).$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} j(u(t,x) - \bar{u}(t,x))\varphi(x)^\alpha &\leqslant 2\alpha \int_{\mathbb{R}^N} |p(u(t,x) - \bar{u}(t,x))| \varphi^{\alpha-1} |\nabla \varphi| \\ &\leqslant 2\alpha \left(\int_{\mathbb{R}^N} (|p(u(t,x) - \bar{u}(t,x))| \varphi^{\alpha-1})^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha} \left(\int_{\mathbb{R}^N} |\nabla \varphi|^\alpha \right)^{1/\alpha} \\ &\leqslant 2\alpha^2 \left(\int_{\mathbb{R}^N} |T_k^+(u(t,x) - \bar{u}(t,x))^\alpha| \varphi^\alpha \right)^{(\alpha-1)/\alpha} \left(\int_{\mathbb{R}^N} |\nabla \varphi|^\alpha \right)^{1/\alpha}. \end{aligned} \tag{40}$$

Now, we observe that $T_k^+(r)^\alpha \leqslant j_\alpha(r)$ for all $r \in \mathbb{R}$. Hence

$$\frac{d}{dt} \int_{\mathbb{R}^N} j(u(t,x) - \bar{u}(t,x))\varphi^\alpha \leqslant 2\alpha^2 \left(\int_{\mathbb{R}^N} j(u(t,x) - \bar{u}(t,x))\varphi^\alpha \right)^{(\alpha-1)/\alpha} \\ \times \left(\int_{\mathbb{R}^N} |\nabla \varphi|^\alpha \right)^{1/\alpha},$$

and, therefore,

$$\frac{d}{dt} \left(\int_{\mathbb{R}^N} j(u(t,x) - \bar{u}(t,x))\varphi^\alpha \right)^{1/\alpha} \leqslant 2\alpha \left(\int_{\mathbb{R}^N} |\nabla \varphi|^\alpha \right)^{1/\alpha}.$$

Setting $\varphi_n(x) := \varphi(\frac{x}{n})$ instead of $\varphi(x)$ we get

$$\begin{aligned} \frac{d}{dt} \left(\int_{\mathbb{R}^N} j(u(t,x) - \bar{u}(t,x))\varphi_n^\alpha \right)^{1/\alpha} &\leqslant 2\alpha \left(\int_{\mathbb{R}^N} |\nabla \varphi_n|^\alpha \right)^{1/\alpha} \\ &= 2\alpha n^{(N-\alpha)/\alpha} \left(\int_{\mathbb{R}^N} |\nabla \varphi|^\alpha \right)^{1/\alpha}. \end{aligned}$$

Integrating from 0 to T and using the facts that $u(t) \to u_0$, $\bar{u}(t) \to \bar{u}_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \to 0^+$, we have

$$\left(\int_{\mathbb{R}^N} j(u(T,x) - \bar{u}(T,x))\varphi_n^\alpha\right)^{1/\alpha} \leqslant \left(\int_{\mathbb{R}^N} j(u_0 - \bar{u}_0)\varphi_n^\alpha\right)^{1/\alpha} + 2\alpha T n^{(N-\alpha)/\alpha}\left(\int_{\mathbb{R}^N} |\nabla\varphi|^\alpha\right)^{1/\alpha}. \quad (41)$$

Letting $n \to \infty$ and recalling that $\alpha > N$, we obtain that

$$\int_{\mathbb{R}^N} j(u(T,x) - \bar{u}(T,x)) \leqslant \int_{\mathbb{R}^N} j(u_0 - \bar{u}_0). \quad \blacksquare$$

COROLLARY 1. *Let $u_0, \bar{u_0} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Let $u, \bar{u}$ be two entropy solutions of* (1) *with initial conditions $u_0, \bar{u_0}$, respectively. If $u_0 \leqslant \bar{u_0}$ then $u \leqslant \bar{u}$. In particular, the entropy solution of* (1) *is unique.*

Proof of the Last Assertion of Theorem 3. Write (41) for $u(t,x)$ and $u_k(t,x)$. We have

$$\left(\int_{\mathbb{R}^N} j(u(t,x) - u_k(t,x))\varphi_n^\alpha\right)^{1/\alpha} \leqslant \left(\int_{\mathbb{R}^N} j(u_0 - u_{0k})\varphi_n^\alpha\right)^{1/\alpha} + 2\alpha t n^{(N-\alpha)/\alpha}\left(\int_{\mathbb{R}^N} |\nabla\varphi|^\alpha\right)^{1/\alpha},$$

for any $t \in [0, T]$ and any $n, k \geqslant 1$. Given $p \in \mathbb{N}$, let $n_p \in \mathbb{N}$ be such that

$$2\alpha T n_p^{(N-\alpha)/\alpha}\left(\int_{\mathbb{R}^N} |\nabla\varphi|^\alpha\right)^{1/\alpha} \leqslant \frac{1}{p}.$$

Choose now $\varphi \in C_0^\infty(\mathbb{R}^N)$ of the form $\varphi(x) = \phi(|x|)$ where ϕ is a decreasing function. By our choice of φ we have that

$$\left(\int_{\mathbb{R}^N} j(u(t,x) - u_k(t,x))\varphi^\alpha\right)^{1/\alpha} \leqslant \left(\int_{\mathbb{R}^N} j(u(t,x) - u_k(t,x))\varphi_{n_p}^\alpha\right)^{1/\alpha} \leqslant \left(\int_{\mathbb{R}^N} j(u_0 - u_{0k})\varphi_{n_p}^\alpha\right)^{1/\alpha} + \frac{1}{p}$$

for any $t \in [0, T]$ and any $k \geqslant 1$. Now, let $k_p \in \mathbb{N}$ be such that

$$\left(\int_{\mathbb{R}^N} j(u_0 - u_{0k})\varphi_{n_p}^\alpha\right)^{1/\alpha} \leqslant \frac{1}{p}$$

for any $k \geqslant k_p$. Then

$$\left(\int_{\mathbb{R}^N} j(u(t,x) - u_k(t,x))\varphi^\alpha \right)^{1/\alpha} \leqslant \frac{2}{p}$$

for any $t \in [0, T]$ and any $k \geqslant k_p$. We conclude that $u_k \to u$ in $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$. ∎

Remark 3. The same proof above yields that (u_k) is a Cauchy sequence in $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ when (u_{0k}) is a Cauchy sequence in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

6. EXISTENCE IN $L^1_{\mathrm{loc}}(\mathbb{R}^N)$

LEMMA 2. *Let $u_0 \in L^2(\mathbb{R}^N)$ and let u be the strong solution of* (1) *and* (2). *Let $T > 0$, $p \in \mathscr{P}$, set $j(r) := \int_0^r p(s)\, ds$, and let $\varphi \in C^\infty([0, T] \times \mathbb{R}^N)$ with compact support in x. Then*

$$\begin{aligned} &\int_{\mathbb{R}^N} j(u(T))\varphi(T) - \int_0^T \int_{\mathbb{R}^N} j(u)\varphi_t + \int_0^T \int_{\mathbb{R}^N} \varphi\, d|D(p(u))| \\ &\quad \leqslant - \int_0^T \int_{\mathbb{R}^N} z \cdot \nabla\varphi\ p(u) + \int_{\mathbb{R}^N} j(u_0)\varphi(0). \end{aligned} \tag{42}$$

If in addition $p \in \mathscr{P} \cap C^1(\mathbb{R})$, then the equality holds in (42). *In particular, u is an entropy solution of* (1).

Proof. Assume first that p is of class C^1. *Then*

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} j(u)\varphi &= \int_{\mathbb{R}^N} p(u)u_t\varphi + \int_{\mathbb{R}^N} j(u)\varphi_t \\ &= - \int_{\mathbb{R}^N} \varphi\, d(z, D(p(u))) - \int_{\mathbb{R}^N} z \cdot \nabla\varphi\ p(u) + \int_{\mathbb{R}^N} j(u)\varphi_t. \end{aligned}$$

Integrating both terms of the above equality in $]0, T[$, and using the fact that

$$\int_{\mathbb{R}^N} \varphi\, d(z(t), D(p(u(t)))) = \int_{\mathbb{R}^N} \varphi\, d|D(p(u(t)))| \qquad \text{for a.e. } t \in]0, T[,$$

which is a consequence of Proposition 2.8 in [8] (here we use $p \in C^1$) and the equality

$$\int_{\mathbb{R}^N} \varphi\, d(z(t), Du(t)) = \int_{\mathbb{R}^N} \varphi\, d|Du(t)| \qquad \text{for a.e. } t \in]0, T[,$$

we obtain

$$\int_{\mathbb{R}^N} j(u(T))\varphi(T) - \int_0^T \int_{\mathbb{R}^N} j(u)\varphi_t + \int_0^T \int_{\mathbb{R}^N} \varphi \, d|D(p(u))| = -\int_0^T \int_{\mathbb{R}^N} z \cdot \nabla\varphi \, p(u) + \int_{\mathbb{R}^N} j(u_0)\varphi(0). \tag{43}$$

If $p \in \mathscr{P}$ is generic, we approximate p in the uniform norm with functions $p_n \in \mathscr{P} \cap C^1(\mathbb{R})$, then write (43) for p_n instead of p and let $n \to \infty$ to conclude that (42) holds. ∎

Proof. Existence. Let $u_0 \in L^1_{\rm loc}(\mathbb{R}^N)$. Let $u_{0n} \in L^2(\mathbb{R}^N)$ be such that $u_{0n} \to u_0$ in $L^1_{\rm loc}(\mathbb{R}^N)$. Let u_n be the strong solutions of (1) corresponding to the initial conditions u_{0n}. By Remark 3, (u_n) is a Cauchy sequence in $C([0, T]; L^1_{\rm loc}(\mathbb{R}^N))$. Thus we may assume that $u_n \to u$ in $C([0, T]; L^1_{\rm loc}(\mathbb{R}^N))$ for some $u \in C([0, T]; L^1_{\rm loc}(\mathbb{R}^N))$. In particular, we have that $u(t) \to u_0$ in $L^1_{\rm loc}(\mathbb{R}^N)$ as $t \to 0+$.

Now, let $p \in \mathscr{P}$ and let $\varphi \in C_0^\infty(]0, T[\times\mathbb{R}^N)$. Inserting $u = u_n$ into (42) gives

$$-\int_0^T \int_{\mathbb{R}^N} j(u_n)\varphi_t + \int_0^T \int_{\mathbb{R}^N} \varphi \, d|D(p(u_n))| \leqslant -\int_0^T \int_{\mathbb{R}^N} z_n \cdot \nabla\varphi \, p(u_n), \tag{44}$$

with an equality if $p \in \mathscr{P} \cap C^1(\mathbb{R})$. In particular, the choice of $j(r) = r$, i.e., $p(r) = 1$, gives

$$\int_0^T \int_{\mathbb{R}^N} u_n\varphi_t = \int_0^T \int_{\mathbb{R}^N} z_n \cdot \nabla\varphi. \tag{45}$$

Possibly passing to a subsequence, we may assume that $z_n \to z$ weakly* in $(L^\infty(]0, T[\times\mathbb{R}^N))^N$. Letting $n \to \infty$ in (45) we have

$$\int_0^T \int_{\mathbb{R}^N} u\varphi_t = \int_0^T \int_{\mathbb{R}^N} z \cdot \nabla\varphi. \tag{46}$$

We conclude $u_t = \operatorname{div} z$ in $\mathscr{D}'(]0, T[\times\mathbb{R}^N)$. As $j(u_n) \to j(u)$ and $p(u_n) \to p(u)$ in $C([0, T]; L^1_{\rm loc}(\mathbb{R}^N))$, letting $n \to \infty$ in (44) we obtain

$$-\int_0^T \int_{\mathbb{R}^N} j(u)\varphi_t + \int_0^T \int_{\mathbb{R}^N} \varphi \, d|D(p(u))| \leqslant -\int_0^T \int_{\mathbb{R}^N} z \cdot \nabla\varphi \, p(u)$$

provided $\varphi \geqslant 0$. In particular, since $j(u), p(u) \in C([0, T]; L^1_{\rm loc}(\mathbb{R}^N))$ we have

$$p(u) \in L^1_w(]0, T[; \mathrm{BV}_{\rm loc}(\mathbb{R}^N)) \qquad \forall p \in \mathscr{P},$$

and we conclude that u is an entropy solution of (1).

7. TIME REGULARITY

Let us recall the basic estimates of semigroups generated by subdifferentials. According to Step 3 of Theorem 2 and [16, Theorem 3.2] (estimate (15) with $v = 0$) and [16, Theorem 3.6] (with $f = 0$, $K = \{0\}$) we have that

$$\operatorname*{ess\,sup}_{s \in]t,\infty[} \int_{\mathbb{R}^N} |u_t(s, x)|^2 \, dx \leqslant \frac{1}{t} \int_{\mathbb{R}^N} |u_0|^2 \, dx \qquad \forall t > 0, \tag{47}$$

$$\int_0^T \int_{\mathbb{R}^N} |u_t(t, x)|^2 t \, dx \, dt \leqslant \frac{1}{2} \int_{\mathbb{R}^N} |u_0|^2 \, dx \tag{48}$$

and if $u_0 \in \mathrm{BV}(\mathbb{R}^N)$

$$\int_0^T \int_{\mathbb{R}^N} |u_t(t, x)|^2 \, dx \, dt \leqslant \int_{\mathbb{R}^N} |Du_0|. \tag{49}$$

Our purpose is to localize estimates (48) and (49). To cover the case of initial conditions in $L^1_{\rm loc}(\mathbb{R}^N)$, we need to consider the family $\mathscr{T} \subseteq \mathscr{P}$ of truncatures $T_{a,b}$, with $a < b$, defined by

$$T_{a,b}(r) = \begin{cases} a & \text{if } r < a, \\ r & \text{if } a \leqslant r \leqslant b, \\ b & \text{if } r > b. \end{cases}$$

PROPOSITION 3. *Let $u_0 \in L^2(\mathbb{R}^N)$ and let u be the strong solution of* (1) *and* (2). *Then*

$$p(u)_t \in L^2_{\rm loc}(]0, T[; L^2(\mathbb{R}^N)), \qquad t^{1/2} p(u)_t \in L^2(]0, T[; L^2(\mathbb{R}^N)), \qquad \forall p \in \mathscr{T}.$$

Moreover, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and any $s < t$ such that $p(u(s)) \in \mathrm{BV}_{\rm loc}(\mathbb{R}^N)$ we have the estimate

$$\begin{aligned} &\frac{1}{2} \int_s^t \int_{\mathbb{R}^N} |p(u)_t|^2 \varphi^2 + \int_{\mathbb{R}^N} \varphi^2 \, d|D(p(u(t)))| \\ &\quad \leqslant \int_{\mathbb{R}^N} \varphi^2 \, d|D(p(u(s)))| + 2(t-s) \int_{\mathbb{R}^N} |\nabla \varphi|^2, \end{aligned} \tag{50}$$

and, if T is such that $u(T) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N)$, also

$$\begin{aligned}\frac{1}{2}\int_0^T \int_{\mathbb{R}^N} t|p(u)_t|^2\varphi^2 + T\int_{\mathbb{R}^N} \varphi^2\, d|Dp(u(T))| \\ \leqslant \int_0^T \int_{\mathbb{R}^N} \varphi^2\, d|Dp(u(t))| + T^2 \int_{\mathbb{R}^N} |\nabla\varphi|^2.\end{aligned} \tag{51}$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and set

$$I := \left\{ s \in]0, T[: u(s) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N) \int_{\mathbb{R}^N} |u_t(s,x)|^2\, dx \leqslant \frac{1}{s}\int_{\mathbb{R}^N} |u_0|^2\, dx \right\}.$$

We recall that $]0, T[\setminus I$ has zero measure. Let $s, t \in I$. Multiply the equation $u_t(t) = \operatorname{div} z(t)$ by $(p(u(t)) - p(u(s)))\varphi^2$ and integrate over $\mathbb{R}^N$. After integrating by parts, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} \varphi^2\, d(|D(p(u(t)))| - |D(p(u(s)))|) \\ &\leqslant \int_{\mathbb{R}^N} u_t(t)[p(u(s)) - p(u(t))]\varphi^2 - \int_{\mathbb{R}^N} z(t)\cdot\nabla\varphi^2[p(u(t)) - p(u(s))].\end{aligned} \tag{52}$$

Let $\delta > 0$ and let $s, t \in I$, $s, t \geqslant \delta$. Using (47), we have

$$\begin{aligned}\int_{\mathbb{R}^N} \varphi^2\, d(|D(p(u(t)))| - |D(p(u(s)))|) \leqslant \frac{1}{\delta}\|u_0\|_2\|[p(u(s)) - p(u(t))]\varphi^2\|_2 \\ + \int_{\mathbb{R}^N} |\nabla\varphi^2||p(u(t)) - p(u(s))|.\end{aligned} \tag{53}$$

Since a similar inequality holds with s and t interchanged, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} \varphi^2\, d(|D(p(u(t)))| - |D[p(u(s))]|)\right| \\ &\leqslant \frac{1}{\delta}\|u_0\|_2\|(p(u(s)) - p(u(t)))\varphi^2\|_2 + \int_{\mathbb{R}^N} |\nabla\varphi^2||p(u(t)) - p(u(s))|.\end{aligned} \tag{54}$$

As $u \in W^{1,2}_{\mathrm{loc}}(]0, T[; L^2(\mathbb{R}^N))$, i.e., is a locally absolutely continuous function of time, then also $p(u)$ is and, from (53), we deduce that $\int_{\mathbb{R}^N} \varphi^2\, d|D(p(u))|$ is absolutely continuous in $]\delta, T[$ for any $\delta > 0$ sufficiently small. Put $s = t - h \in I$ in (52), divide by $h > 0$, and let $h \to 0^+$. We obtain, at any differentiability point t of u and $\int_{\mathbb{R}^N} \varphi^2\, d|D(p(u))|$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} p'(u)u_t^2\varphi^2 + \frac{d}{dt}\int_{\mathbb{R}^N} \varphi^2\, d|D(p(u))| \\
&\quad\leqslant 2\int_{\mathbb{R}^N} |p(u)_t||\varphi||\nabla\varphi| \\
&\quad\leqslant 2\left(\int_{\mathbb{R}^N} |p(u)_t|^2\varphi^2\right)^{1/2}\left(\int_{\mathbb{R}^N} |\nabla\varphi|^2\right)^{1/2} \\
&\quad\leqslant \frac{1}{2}\int_{\mathbb{R}^N} |p(u)_t|^2\varphi^2 + 2\|\nabla\varphi\|_2^2.
\end{aligned}$$

Since $p'(r) \in \{0, 1\}$ for almost every r, we have

$$\frac{1}{2}\int_{\mathbb{R}^N} |p(u)_t|^2\varphi^2 + \frac{d}{dt}\int_{\mathbb{R}^N} \varphi^2\, d|D(p(u))| \leqslant 2\|\nabla\varphi\|_2^2. \tag{55}$$

Observe that inequality (55) holds almost everywhere in $]0, T[$. Choosing $s \in I$ and integrating (55) in $]s, t[$ we obtain (50). Since φ does not depend on time, from (42) it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} j(u(T))\varphi^2 + \int_0^T \int_{\mathbb{R}^N} \varphi^2\, d|D(p(u))| \\
&\qquad\leqslant \int_0^T \int_{\mathbb{R}^N} |\nabla\varphi^2||p(u)| + \int_0^T \int_{\mathbb{R}^N} j(u_0)\varphi^2.
\end{aligned} \tag{56}$$

Inequality (56) proves that $\int_{\mathbb{R}^N} \varphi^2\, d|D(p(u))| \in L^1(0, T)$. Hence $t_n \int_{\mathbb{R}^N} \varphi^2 |D(p(u(t_n)))| \to 0$ for a subsequence $t_n \to 0+$, $t_n \in I$. Multiplying (55) by t and integrating on $]t_n, T[$ we obtain

$$\frac{1}{2}\int_{t_n}^T \int_{\mathbb{R}^N} t|p(u)_t|^2\varphi^2 + \int_{t_n}^T t\frac{d}{dt}\int_{\mathbb{R}^N} \varphi^2\, d|D(p(u))| \leqslant (T^2 - t_n^2)\int_{\mathbb{R}^N} |\nabla\varphi^2|.$$

Integrating by parts with respect to time we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{t_n}^T \int_{\mathbb{R}^N} t|p(u)_t|^2\varphi^2 + T\int_{\mathbb{R}^N} \varphi^2\, d|D(p(u(T)))| \\
&\quad\leqslant \int_{t_n}^T \int_{\mathbb{R}^N} \varphi^2\, d|D(p(u))| + t_n\int_{\mathbb{R}^N} \varphi^2\, d|D(p(u(t_n)))| + (T^2 - t_n^2)\int_{\mathbb{R}^N} |\nabla\varphi^2|.
\end{aligned}$$

Letting $n \to \infty$, we obtain (51). ∎

COROLLARY 2. *Let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Let u be the entropy solution of* (1) *and* (2). *Then*

$$p(u)_t \in L^2_{\mathrm{loc}}(0, \infty; L^2_{\mathrm{loc}}(\mathbb{R}^N)), \qquad t^{1/2} p(u)_t \in L^2_{\mathrm{loc}}([0, \infty[; L^2_{\mathrm{loc}}(\mathbb{R}^N)), \qquad \forall p \in \mathcal{T}.$$

Proof. Let $(u_{0n}) \subset L^2(\mathbb{R}^N)$ be a sequence such that $u_{0n} \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Let u_n be the strong solution of (1) corresponding to the initial condition u_{0n}. Inserting $u = u_n$ into (42) and using the fact that the corresponding vector fields z_n satisfy $\|z_n\|_\infty \leqslant 1$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} j(u_n(T))\varphi^2 &+ \int_0^T \int_{\mathbb{R}^N} \varphi^2 \, d|D(p(u_n))| \\ &\leqslant \int_0^T \int_{\mathbb{R}^N} |\nabla \varphi^2| |p(u_n)| + \int_{\mathbb{R}^N} j(u_n(0))\varphi^2 \end{aligned} \tag{57}$$

for any $p \in \mathcal{P}$, $T > 0$, $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $n \in \mathbb{N}$. Since the right-hand side of (57) is bounded by

$$C := \|p\|_\infty T \int_{\mathbb{R}^N} |\nabla \varphi^2| \, dx + \sup_n \int_{\mathbb{R}^N} j(u_n(0))\varphi^2,$$

we have

$$\int_0^T \int_{\mathbb{R}^N} \varphi^2 \, d|D(p(u_n))| \leqslant C. \tag{58}$$

Choose now $T > 0$ such that $u_n(T) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N))$ for all n. Using (51) and (58) we have

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^N} t|p(u_n)_t|^2 \varphi^2 \leqslant C + T^2 \int_{\mathbb{R}^N} |\nabla \varphi|^2. \tag{59}$$

Since $p(u_n) \to p(u)$ in $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$, letting $n \to \infty$ in (59) we obtain

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^N} t p(u)_t^2 \varphi^2 \leqslant C + T^2 \int_{\mathbb{R}^N} |\nabla \varphi|^2.$$

Since this holds for almost every $T > 0$, the conclusion follows. ∎

Remark 4. If $p(u_0) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N)$ we have

$$\begin{aligned} &p(u) \in L^1_w(]0, T[; \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N)), \\ &\quad p(u) \in W^{1,2}(]0, T[; L^2_{\mathrm{loc}}(\mathbb{R}^N)) \subseteq C([0, T]; L^2_{\mathrm{loc}}(\mathbb{R}^N)) \end{aligned}$$

for any $p \in \mathcal{T}$. Indeed, this follows from (50) instead of using (51) in the above argument.

If u is the entropy solution of (1) and (2) for $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $K \in \mathbb{R}$, then $v(t) := u(t) + K$ is the entropy solution of (1) whose initial condition is $v(0) = u_0 + K$. If we denote by $S(t)$ the semigroup in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ constructed from the entropy solutions, we may write $S(t)(u_0 + K) = S(t)u_0 + K$ for any $u(0) = u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $K \in \mathbb{R}$.

PROPOSITION 4. *Let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ with $u_0 \geqslant -M$ for some $M > 0$. If u is the entropy solution of* (1) and (2) *we have*

$$u'(t) \leqslant \frac{u(t) + M}{t} \qquad \textit{for a.e. } t > 0.$$

Moreover, $u_t \in L^1_{\mathrm{loc}}(]0, T[; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ for any $T > 0$. A similar statement holds if $u_0 \leqslant M$ for some $M > 0$.

Proof. Let $0 \leqslant v_{0n} \in L^2(\mathbb{R}^N)$ be such that $v_{0n} \to u_0 + M$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Let $v_n(t) := S(t)(v_{0n})$. By Proposition 1 we have

$$v'_{nt} \leqslant \frac{v_n}{t} \qquad \text{for a.e. } t > 0.$$

Since $v_n(t) = S(t)(v_{0n}) \to S(t)(u_0 + M) = S(t)(u_0) + M = u(t) + M$ in $L^1(]0, T[; L^1_{\mathrm{loc}}(\mathbb{R}^N))$, it follows that

$$u_t \leqslant \frac{u + M}{t} \qquad \text{in } \mathcal{D}'(]0, T[\times \mathbb{R}^N). \tag{60}$$

By estimate (60), u_t is a Radon measure in $]s, t[\times \mathbb{R}^N$, for all $0 < s < t$ and $R > 0$. Thus

$$\int_s^t \int_{B_R(0)} |u_t| < \infty. \tag{61}$$

in any ball $B_R(0)$, $R > 0$. Now, taking $p = T_{ab}$, the estimate in Corollary 2 says that u_t is a function in $L^2(Q_{a,b} \cap B_R(0))$, for all $a < b$, where $Q_{a,b} := \{(t, x) \in Q\colon a < u(t, x) < b\}$, and all $R > 0$. This observation together with (61) proves that $u_t \in L^1_{\mathrm{loc}}(]0, T[; L^1_{\mathrm{loc}}(\mathbb{R}^N))$. ∎

We conclude this section with the following observation. The existence and uniqueness results for (1) and (2) may be used to prove an estimate for the time derivative of the solution of

$$\frac{\partial v}{\partial t} = \operatorname{div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right) \qquad \text{in }]0,\infty[\times\mathbb{R}^N, \tag{62}$$

when the initial datum $v(0,x) = v_0(x) \in L^1(\mathbb{R}^N)$. First, we observe that existence and uniqueness results for (62) when $v_0 \in L^1(\mathbb{R}^N)$ can be obtained following the approach in [6]. Next, we notice that if v is the solution of (62) corresponding to the initial condition $v_0 \in L^1(\mathbb{R}^N)$, then $u(t,x,x_{N+1}) = v(t,x) - x_{N+1}$ is the entropy solution of (1) in $\mathbb{R}^{N+1}$ such that $u(0,x,x_{N+1}) = v_0(x) - x_{N+1}$. In other words, the semigroups $T(t)$ and $S(t)$ associated with (62) and (1) satisfy

$$S(t)(v_0 - x_{N+1}) = T(t)v_0 - x_{N+1} \qquad \text{for any } v_0 \in L^1(\mathbb{R}^N).$$

Now, proceeding as in the proof of Proposition 1 with $\lambda = \frac{t+h}{t}$ we obtain

$$\begin{aligned} v(t+h) - v(t) &= u(t+h) - u(t) = \frac{h}{t+h}\, u(t+h) + S(t)(\lambda^{-1}(v_0 - x_{N+1})) - u(t) \\ &= \frac{h}{t+h}\, u(t+h) + T(t)(\lambda^{-1}v_0) - T(t)v_0 + \frac{h}{t+h}\, x_{N+1} \\ &= \frac{h}{t+h}\, v(t+h) + T(t)(\lambda^{-1}v_0) - T(t)v_0. \end{aligned}$$

This implies that

$$\left\|\frac{v(t+h)-v(t)}{h}\right\|_1 \leqslant \frac{2}{t+h}\|v_0\|_1.$$

From this, and using the techniques of completely accretive operators [15] as in [3] it can be proved that $\|v_t\|_1 \leqslant \frac{2}{t}\|v_0\|_1$.

8. EVOLUTION OF SETS IN $\mathbb{R}^2$: THE CONNECTED CASE

Throughout this section, as well as in Sections 9–11, we take $N = 2$. Let $B \subset \mathbb{R}^2$ be an open set; we say that ∂B is of class $C^{1,1}$ if ∂B can be written, locally around each point, as the graph (with respect to a suitable orthogonal coordinate system) of a function f of class C^1 with Lipschitz continuous gradient, and B can be written (locally) as the epigraph of f. If ∂B is of class $C^{1,1}$, we denote by $\kappa_{\partial B}$ the ($\mathscr{H}^1$-almost everywhere defined) curvature of ∂B.

Let $\Omega \subset \mathbb{R}^2$ be a bounded set of finite perimeter. We set

$$\lambda_\Omega := \frac{P(\Omega)}{|\Omega|}.$$

We want to study when the function

$$u(t,x) := (1 - \lambda_\Omega t)^+ \chi_\Omega(x) \tag{63}$$

is the entropy solution of (1) and (2) when we choose $u_0 = \chi_\Omega$.

Remark 5. The function u defined in (63) is the solution of (1) and (2) with $u(0,x) = \chi_\Omega(x)$ if and only if the function $v := \chi_\Omega$ satisfies the equation

$$-\mathrm{div}\left(\frac{Dv}{|Dv|}\right) = \lambda_\Omega v, \tag{64}$$

i.e., if and only if there exists a vector field $\xi \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ such that $\|\xi\|_\infty \leqslant 1$,

$$-\mathrm{div}\,\xi = \lambda_\Omega v \tag{65}$$

and

$$\int_{\mathbb{R}^2} (\xi, Dv) = \int_{\mathbb{R}^2} |Dv|. \tag{66}$$

With a little abuse of notation, we also write that the pair (v,ξ) is a solution of (64).

It is clear that if v is a solution of (64) then $\lambda_\Omega v$ is a solution of (4).

If χ_Ω is a solution of (64) and C is a connected component of Ω, using (65) and (66) it follows that

$$\lambda_C = \lambda_\Omega. \tag{67}$$

Definition 3. Let $\Omega \subseteq \mathbb{R}^2$ be a set of finite perimeter. We say that Ω is $-$ calibrable if there exists a vector field $\xi_\Omega^- : \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties:

(i) $\xi_\Omega^- \in L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ and $\mathrm{div}\,\xi_\Omega^- \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$;
(ii) $|\xi_\Omega^-| \leqslant 1$ almost everywhere in Ω;
(iii) $\mathrm{div}\,\xi_\Omega^-$ is constant on Ω;
(iv) $\theta(\xi_\Omega^-, -D\chi_\Omega)(x) = -1$ for $\mathscr{H}^1$-almost every $x \in \partial^*\Omega$.

We say that Ω is +calibrable if there exists a vector field $\xi_\Omega^+ : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying properties (i), (ii), (iii), and such that $\theta(\xi_\Omega^+, -D\chi_\Omega)(x) = 1$ for $\mathscr{H}^1$-almost every $x \in \partial^*\Omega$.

Heuristically, condition (iv) says that the inner (resp. outer) normal trace of ξ_Ω^- (resp. of ξ_Ω^+) is 1.

It is clear that Ω is $-$ calibrable if and only if Ω is $+$ calibrable (it is sufficient to define $\xi_\Omega^+ := -\xi_\Omega^-$). Moreover, if Ω is bounded and $-$ calibrable, the constant in (iii) equals $-\lambda_\Omega$, i.e., $-\mathrm{div}\, \xi_\Omega^- \equiv \lambda_\Omega$ on Ω.

The following remark should be compared with (a) of Proposition 5.

Remark 6. Let $\Omega \subset \mathbb{R}^2$ be a bounded set of finite perimeter which is $-$ calibrable. Then

$$\frac{P(\Omega)}{|\Omega|} \leqslant \frac{P(D)}{|D|} \qquad \forall D \subseteq \Omega, \quad D \text{ of finite perimeter.} \tag{68}$$

Indeed,

$$\lambda_\Omega = \frac{1}{|D|} \int_D -\mathrm{div}\, \xi_\Omega^- \, dx \leqslant \frac{1}{|D|} P(D).$$

Remark 7. Let $\Omega \subset \mathbb{R}^2$ be a bounded set of finite perimeter. Assume that Ω is $-$ calibrable and that $\mathbb{R}^2 \setminus \Omega$ is $+$ calibrable. Define

$$\xi := \begin{cases} \xi_\Omega^- & \text{on } \Omega, \\ \xi_{\mathbb{R}^2\setminus\Omega}^+ & \text{on } \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Then $\xi \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ and $\mathrm{div}\, \xi \in L^\infty(\mathbb{R}^2)$.

LEMMA 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set of finite perimeter. Then $v := \chi_\Omega$ is a solution of* (64) *if and only if Ω is $-$ calibrable with $-\mathrm{div}\, \xi_\Omega^- = \lambda_\Omega$ in Ω and $\mathbb{R}^2 \setminus \Omega$ is $+$ calibrable, with $\mathrm{div}\, \xi_{\mathbb{R}^2\setminus\Omega}^+ = 0$ in $\mathbb{R}^2 \setminus \Omega$.*

Proof. If (χ_Ω, ξ) is a solution of (64), then $\xi_\Omega^- := \xi$, $\xi_{\mathbb{R}^2\setminus\Omega}^+ := \xi$ satisfy (i)–(iii) of Definition 3. Moreover, by (66) and (12)

$$\int_{\partial^*\Omega} \theta(\xi_\Omega^-, D\chi_\Omega)\, d\mathscr{H}^1 = P(\Omega) = \int_{\partial^*\Omega} \theta(\xi_{\mathbb{R}^2\setminus\Omega}^+, -D\chi_{\mathbb{R}^2\setminus\Omega})\, d\mathscr{H}^1,$$

so that (iv) of Definition 3 is satisfied. Conversely, it is enough to define $\xi := \xi_\Omega^- \chi_\Omega + \xi_{\mathbb{R}^2\setminus\Omega}^+ \chi_{\mathbb{R}^2\setminus\Omega}$, and to use Remark 7 to check that (χ_Ω, ξ) solves (64). ∎

We are precisely interested in characterizing the sets of Lemma 3. The following theorem answers to this question, under the additional assumption that Ω is connected; thanks to Remark 5, we can characterize those sets Ω such that the function u in (63) is the solution of (1) and (2) with $u_0 = \chi_\Omega$. In Theorems 6 and 7 of Section 9 we consider the general situation.

THEOREM 4. *Let $C \subset \mathbb{R}^2$ be a bounded set of finite perimeter, and assume that C is connected. The function $v := \chi_C$ is a solution of* (64) *if and only if the following three conditions hold*:

(i) *C is convex*;
(ii) *∂C is of class $C^{1,1}$*;
(iii) *the following inequality holds*:

$$\operatorname{ess}\sup_{p\in\partial C} \kappa_{\partial C}(p) \leqslant \frac{P(C)}{|C|}. \tag{69}$$

To prove Theorem 4, we need several intermediate steps. We start with the proof of the implication

$$\chi_C \text{ solution of (64) } \Rightarrow \text{(i)}-\text{(iii) hold}, \tag{70}$$

which will be given after Lemma 7.

Given any set $D \subseteq \mathbb{R}^2$, we define

$$D_\rho := \bigcup \{B_\rho\colon B_\rho \text{ open ball of radius } \rho \text{ contained in } C\},$$

where $\rho > 0$ is small enough such that D_ρ is nonempty.

The result of the next lemma, without an estimate on the curvature, is proved in [28, Proposition 2.4.3]. Since in the following the estimate on the curvature plays a crucial role, we need to include the proof.

LEMMA 4. *Let $C \subset \mathbb{R}^2$ be a bounded open convex set. The following conditions are equivalent*:

(a) *there exists $\rho > 0$ such that $C = C_\rho$*;
(b) *∂C is of class $C^{1,1}$ and* $\operatorname{ess}\sup_{p\in\partial C} \kappa_{\partial C}(p) \leqslant \frac{1}{\rho}$.

Proof. (a) $\Rightarrow$ (b): Assume that $C = C_\rho$ for some $\rho > 0$ and fix a point $z \in \partial C$. Up to a translation and rotation of coordinates, we can suppose that $z = 0$, that ∂C can be written, in a neighborhood of 0, as the graph Γ_f, with respect to the x-variable, of a nonnegative convex function f vanishing at 0 (therefore the open epigraph of f coincides with C in a neighborhood of z). Since $C = C_\rho$, the open ball of radius ρ contained in the epigraph of f and tangent to Γ_f at $(0,0)$ lies locally above f. Therefore we can choose a parabola tangent to Γ_f at $(0,0)$, lying locally inside the epigraph of f and above the ball, whose graph has curvature at zero equals $\frac{1}{\rho} + \varepsilon$. Precisely, for any $\varepsilon > 0$ sufficiently small there exists $\delta > 0$ such that $f(x) \leqslant (\frac{1}{2\rho} + \varepsilon)x^2$ for any $|x| \leqslant \delta$. It follows that f is differentiable at $x = 0$ with $f'(0) = 0$, i.e., ∂C is differentiable at z. Therefore ∂C is differentiable at any point. Since ∂C is convex and differentiable at any point, it follows that ∂C is of class C^1.

Let us now prove that ∂C is of class $C^{1,1}$. The idea is the same as before, but now we need a family of parabolas locally above f, passing to an arbitrary point $(t, f(t))$ for $|t| \leqslant \delta$ and tangent (at the same point) to Γ_f. It will follow that ∂C is locally an infimum of parabolas with second derivative larger than $\frac{1}{\rho}$ (up to ε). Precisely, as $C = C_\rho$, given $\varepsilon > 0$ sufficiently small and possibly reducing δ, we have

$$f(x) \leqslant \phi_t(x) := \left(\frac{1}{2\rho} + \varepsilon\right)(x - a(t))^2 + b(t) \qquad \forall |x|,\ |t| \leqslant \delta,$$

where $a(t) := t - \frac{f'(t)}{(1/\rho)+2\varepsilon}$ and $b(t) := f(t) - \frac{f'(t)^2}{(2/\rho)+4\varepsilon}$ (note that $f \in C^1$, so that a and b are well defined). Since

$$f = \inf_{|t| \leqslant \delta} \phi_t \qquad \text{on } |x| \leqslant \delta,$$

and since ϕ_t are semiconcave with semiconcavity constant equal to $\frac{1}{2\rho} + \varepsilon$ for any $|t| \leqslant \delta$, it follows that f is semiconcave on $[-\delta, \delta]$ with semiconcavity constant equal to $\frac{1}{2\rho} + \varepsilon$. Hence f is of class $C^{1,1}$ in $[-\delta, \delta]$ and $f'' \leqslant \frac{1}{\rho} + \frac{\varepsilon}{2}$ almost everywhere in $[-\delta, \delta]$. Therefore ∂C is of class $C^{1,1}$ and, since ε is arbitrary, $\operatorname{ess\,sup}_{p \in \partial C} \kappa_{\partial C}(p) \leqslant \frac{1}{\rho}$.

The implication (b) $\Rightarrow$ (a) is a particular case of [11, Lemma 9.2] with the choices $P = C$, $\tilde{\phi}(\xi_1, \xi_2) = \sqrt{\xi_1^2 + \xi_2^2}$ and $\lambda = \rho$. ∎

Remark 8. If condition (a) of Lemma 4 holds, then $C = C_\sigma$ for any $\sigma \in [0, \rho]$, since any ball B_ρ of radius ρ is the union of all balls B_σ of radius $\sigma \in [0, \rho]$ contained in B_ρ.

LEMMA 5. *Let $a, b \in \mathbb{R}$, $a < b$, $\lambda > 0$ and $G_\lambda : H_0^1([a,b]) \to \mathbb{R}$ be defined as*

$$G_\lambda(u) := \int_{[a,b]} [\sqrt{1 + (u'(s))^2} - \lambda u(s)]\, d\mathcal{H}^1(s). \tag{71}$$

Assume that there exists a function $u_\lambda \in H_0^1([a,b])$ whose graph is contained in a translated of $\partial B_{1/\lambda}$. Then u_λ is the unique minimizer of G_λ in $H_0^1([a,b])$.

Proof. It is a particular case of [11, Lemma 8.4] with the choice $\tilde{\phi}(\xi_1, \xi_2) = \sqrt{\xi_1^2 + \xi_2^2}$. ∎

LEMMA 6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set of finite perimeter. Assume that $\mathbb{R}^2 \backslash \Omega$ is + calibrable. Then* $\operatorname{div} \xi^+_{\mathbb{R}^2\backslash\Omega} = 0$ *on* $\mathbb{R}^2\backslash\Omega$.

Proof. Let for simplicity $\xi := \xi^+_{\mathbb{R}^2\backslash\Omega}$. Let $R > 0$ be such that $B_R \supseteq \Omega$ and let U be the unbounded component of $\mathbb{R}^2\backslash\Omega$. By assumption we have that $\operatorname{div} \xi = \alpha$ on $U \cap B_R$ for some real constant α. Using (12) and the properties of ξ (see (ii) and (iv) of Definition 3) we have

$$-2\pi R + P(U) \leqslant \int_{U \cap B_R} \operatorname{div} \xi \, dx \leqslant 2\pi R + P(U).$$

If we denote by λ the (finite) measure of the union of all connected components of $\mathbb{R}^2\backslash\Omega$ contained in B_R, it follows that

$$\frac{-2\pi R + P(U)}{\pi R^2 - |\Omega| - \lambda} \leqslant \alpha = \frac{\int_{U \cap B_R} \operatorname{div} \xi \, dx}{|U \cap B_R|} \leqslant \frac{2\pi R + P(U)}{\pi R^2 - |\Omega| - \lambda} v.$$

Letting $R \to +\infty$ we deduce $\alpha = 0$. ∎

PROPOSITION 5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set of finite perimeter which is − calibrable and such that $\mathbb{R}^2\backslash\Omega$ is + calibrable. Then*

(a) *the following relations hold*:

$$\frac{P(\Omega)}{|\Omega|} \leqslant \frac{P(D)}{|\Omega \cap D|} \qquad \forall D \subseteq \mathbb{R}^2, \quad D \textit{ of finite perimeter}; \tag{72}$$

(b) *each connected component of Ω is convex.*

Proof. Let $\xi \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$, $\|\xi\|_\infty \leqslant 1$ be the vector field defined by $\xi := \xi^-_\Omega \chi_\Omega + \xi^+_{\mathbb{R}^2\backslash\Omega} \chi_{\mathbb{R}^2\backslash\Omega}$. By Remark 7 we have that $\operatorname{div} \xi \in L^\infty(\mathbb{R}^2)$. Let $D \subseteq \mathbb{R}^2$ be a set of finite perimeter. Using Lemma 6 and the fact that $-\operatorname{div} \xi^-_\Omega \equiv \lambda_\Omega$ on Ω, we have

$$-\int_{\mathbb{R}^2} \chi_D \operatorname{div} \xi \, dx = -\int_{\mathbb{R}^2} \chi_\Omega \chi_D \operatorname{div} \xi \, dx = \lambda_\Omega \int_{\mathbb{R}^2} \chi_{\Omega \cap D} \, dx = \lambda_\Omega |\Omega \cap D|.$$

Hence

$$\lambda_\Omega |\Omega \cap D| \leqslant P(D), \tag{73}$$

and (72) follows.

Moreover from (73) it follows that

$$P(\Omega) \leqslant P(D) \qquad \forall D \supseteq \Omega, \quad D \text{ of finite perimeter.}$$

We conclude that each connected component of Ω must be convex. ∎

Definition 4. Given $\lambda \in \mathbb{R}$ we define the functional $\mathscr{G}_\lambda$ as

$$\mathscr{G}_\lambda(D) := P(D) - \lambda|D|, \qquad D \subseteq \mathbb{R}^2, \quad D \text{ of finite perimeter.}$$

PROPOSITION 6. *Let C be a bounded open convex set, and assume that C is $-$ calibrable. Then ∂C is of class $C^{1,1}$.*

Proof. Set for simplicity $\xi := -\xi_C^-$ and recall that $\operatorname{div} \xi = \lambda_C$ on C. For any $\lambda > \lambda_C$ and any finite perimeter set B strictly contained in C we then have

$$\mathscr{G}_\lambda(B) \geqslant \int_B (\operatorname{div} \xi - \lambda)\, dx > \int_C (\operatorname{div} \xi - \lambda)\, dx = \mathscr{G}_\lambda(C). \tag{74}$$

Assume now by contradiction that ∂C is not of class $C^{1,1}$. By Lemma 4 it follows that C_ρ is strictly contained in C for some $\rho > 0$. Fix $\sigma < \rho$ such that $\sigma\lambda_C < 1$. By Remark 8 we have that C_σ is strictly contained in C. Applying Lemma 5 to the connected components of $\partial C_\sigma \backslash \partial C$, we get

$$\mathscr{G}_{1/\sigma}(C_\sigma) \leqslant \mathscr{G}_{1/\sigma}(C),$$

which contradicts (74). ∎

Remark 9. (i) If $\Omega \subset \mathbb{R}^2$ is a bounded set of finite perimeter satisfying (68) it follows that $\mathscr{G}_{\lambda_\Omega}(D) \geqslant 0$ for any $D \subseteq \Omega$ of finite perimeter, while obviously $\mathscr{G}_{\lambda_\Omega}(\Omega) = 0$. Therefore Ω minimizes $\mathscr{G}_{\lambda_\Omega}$ among all finite perimeter sets $D \subseteq \Omega$.

(ii) By the proof of Proposition 6, it follows that if C is a bounded open convex set which is $-$ calibrable, then C minimizes $\mathscr{G}_\lambda$ among all finite perimeter sets $B \subseteq C$ and where $\lambda > \lambda_C$.

In order to prove implication (70) of Theorem 4 we need one more lemma.

LEMMA 7. *Let $C \subset \mathbb{R}^2$ be a bounded open convex set with $C^{1,1}$ boundary satisfying* (68) *with C in place of Ω. Then* (69) *holds.*

Proof. Let U be a neighborhood of ∂C and let $h \in C_0^1(U)$. Let $\alpha \in \mathbb{R}$ be sufficiently small, and let $\Psi_\alpha(x, y) := (x, y) + \alpha h(x, y)\nu(x, y)$, where $\nu \in C^1(U; \mathbb{R}^2)$ is a vector field satisfying $|\nu| = 1$ on U, and $\nu = \nu^C$ on ∂C. Extend Ψ_α as

$\Psi_\alpha(x, y) = (x, y)$ outside U. Let $C_\alpha \coloneqq \Psi_\alpha(C)$. By Remark 9 it follows that C minimizes $\mathscr{G}_{\lambda_C}$ among all finite perimeter sets contained in C. Therefore, if h is nonpositive,

$$0 \leqslant \lim_{\alpha \to 0^+} \frac{\mathscr{G}_{\lambda_C}(C_\alpha) - \mathscr{G}_{\lambda_C}(C)}{\alpha} = \int_{\partial C} [\kappa_{\partial C} - \lambda_C] h \, d\mathscr{H}^1.$$

It follows $\kappa_{\partial C}(x) \leqslant \lambda_C$ for $\mathscr{H}^1$-almost every $x \in \partial C$. ∎

We are now in the position to prove implication (70) of Theorem 4. If χ_C is a solution of (64), by Lemma 3 (applied with $\Omega = C$) it follows that C is $-$ calibrable with $-\mathrm{div}\, \xi_C^- = \lambda_C$ in C and $\mathbb{R}^2 \setminus C$ is $+$ calibrable with $\mathrm{div}\, \xi^+_{\mathbb{R}^2 \setminus C} = 0$ in $\mathbb{R}^2 \setminus C$. Therefore by Proposition 5 (b) (applied with $\Omega = C$) and the assumption that C is connected it follows that C is convex. Hence by Proposition 6 we have that ∂C is of class $C^{1,1}$. Moreover, inequality (68) holds. Therefore we can apply Lemma 7 to conclude that (69) holds.

Let us now prove the opposite implication of Theorem 4, that is

$$\text{(i)}-\text{(iii)} \Rightarrow \chi_C \text{ solution of (64).} \tag{75}$$

Assume that C is a bounded open $C^{1,1}$ convex set satisfying (69). It has been proved in [27] that (69) is a necessary and sufficient condition for C to be a minimizer of the functional $\mathscr{G}_{\lambda_C}$ among all sets of finite perimeter $D \subseteq C$. In this case the function $f \coloneqq \lambda_C \chi_C$ satisfies $\|f\|_* \leqslant 1$. Indeed, if $w \in L^2(\mathbb{R}^2) \cap \mathrm{BV}(\mathbb{R}^2)$ is nonnegative, we have

$$\begin{aligned} \int_{\mathbb{R}^2} f(x) w(x)\, dx &= \int_0^\infty \int_{\mathbb{R}^2} \lambda_C \chi_C \chi_{\{w \geqslant t\}}\, dx\, dt = \int_0^\infty \lambda_C |C \cap \{w \geqslant t\}|\, dt \\ &\leqslant \int_0^\infty P(C \cap \{w \geqslant t\})\, dt \leqslant \int_0^\infty P(\{w \geqslant t\})\, dt = \int_{\mathbb{R}^2} |Dw|, \end{aligned}$$

where we have used that for all $t \geqslant 0$ for which $\{w \geqslant t\}$ is a set of finite perimeter we have that

$$P(C \cap \{w \geqslant t\}) \leqslant P(\{w \geqslant t\}),$$

which is a consequence of the convexity of C. Splitting any function $\omega \in L^2(\mathbb{R}^2) \cap \mathrm{BV}(\mathbb{R}^2)$ into its positive and negative part, using the above inequality one can prove that $|\int_{\mathbb{R}^2} f(x)\omega(x)\, dx| \leqslant \int_{\mathbb{R}^2} |D\omega|$. It follows that $\|f\|_* \leqslant 1$. Then, by Lemma 1, there is a vector field $\xi \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ with $\|\xi\|_\infty \leqslant 1$ such that

$$-\mathrm{div}\, \xi = f = \lambda_C \chi_C. \tag{76}$$

Now, multiplying (76) by χ_C and integrating by parts, we obtain

$$\int_{\mathbb{R}^2} (\xi, D\chi_C) = \lambda_C \int_{\mathbb{R}^2} \chi_C \, dx = P(C) = \int_{\mathbb{R}^2} |D\chi_C|,$$

hence χ_C is a solution of (64). The proof of Theorem 4 is concluded.

We conclude this section by recalling that in paper [27], condition (69) was used as a necessary and sufficient condition for the existence of a solution u with $\nabla u \in L^\infty_{\mathrm{loc}}(C; \mathbb{R}^2)$ of the equation

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \lambda_C \qquad \text{in } C \tag{77}$$

with boundary condition $\lim_{C \ni y \to x} \frac{\nabla u(y)}{\sqrt{1+|\nabla u(y)|^2}} = -\nu^C(x)$ for any $x \in \partial C$.

9. EVOLUTION OF SETS IN $\mathbb{R}^2$: THE NONCONNECTED CASE

The aim of this section is to generalize Theorem 4 to nonconnected sets (see Theorems 6 and 7). Theorem 7 is basically a further generalization of Theorem 6, and has a self-contained and independent proof. We begin with the following result.

THEOREM 5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and assume that $\partial\Omega$ is of class $C^{1,1}$. Then $\mathbb{R}^2 \backslash \bar{\Omega}$ is + calibrable if and only if*

$$2P(D, \mathbb{R}^2 \backslash \bar{\Omega}) \geqslant P(D), \quad D \subset \mathbb{R}^2 \backslash \bar{\Omega}, \quad D \textit{ bounded of finite perimeter}. \tag{78}$$

Proof. Assume first that $\mathbb{R}^2 \backslash \bar{\Omega}$ is + calibrable and set $\xi := \xi^+_{\mathbb{R}^2 \backslash \bar{\Omega}}$. By Lemma 6 we have $\mathrm{div}\, \xi = 0$ on $\mathbb{R}^2 \backslash \bar{\Omega}$. Let $D \subset \mathbb{R}^2 \backslash \bar{\Omega}$ be a bounded set of finite perimeter. Then

$$0 = \int_D \mathrm{div}\, \xi \, dx \geqslant \mathscr{H}^1(\partial^* D \cap \partial\Omega) - P(D, \mathbb{R}^2 \backslash \bar{\Omega}),$$

which implies (78), since $\mathscr{H}^1(\partial^* D \cap \partial\Omega) = P(D) - P(D, \mathbb{R}^2 \backslash \bar{\Omega})$.

Assume now that (78) holds. Let $R > 0$ be such $B_R := B_R(0) \supseteq \Omega$ and

$$\mathrm{dist}(\partial B_R, \partial\Omega) > \tfrac{1}{2} P(\Omega) \tag{79}$$

and set

$$c := -\frac{P(\Omega)}{2\pi R}. \tag{80}$$

Possibly increasing R, we can assume that $|c| < 1$. Given a bounded open set $A \subset \mathbb{R}^2$ we now define the functional

$$\mathscr{F}(\xi, A) := \int_A (\operatorname{div} \xi)^2 \, dx, \qquad \xi \in H^{\operatorname{div}}(A), \tag{81}$$

where $H^{\operatorname{div}}(A) := \{\xi \in L^2(A; \mathbb{R}^2): \ \operatorname{div} \xi \in L^2(A)\}$. Reasoning as in [9, Proposition 6.1] one can prove that the variational problem

$$\min\{\mathscr{F}(\xi, B_R \setminus \bar{\Omega}): \quad \xi \in H^{\operatorname{div}}(B_R \setminus \bar{\Omega}), \ |\xi| \leqslant 1 \text{ a.e. in } B_R \setminus \bar{\Omega}, \tag{82}$$

$$\theta(\xi, -D\chi_\Omega) = -1 \text{ on } \partial\Omega, \ \theta(\xi, -D\chi_{B_R}) = c \text{ on } \partial B_R\} \tag{83}$$

admits a solution and, if ξ_1 and ξ_2 are two solutions, then $\operatorname{div} \xi_1 = \operatorname{div} \xi_2$ almost everywhere on $B_R \setminus \bar{\Omega}$. Moreover, arguing as in [9, Theorem 6.7; 10, Proposition 3.5, Theorem 5.3], it follows that given any minimizer $\xi_{\min}$ we have $\operatorname{div} \xi_{\min} \in L^\infty(B_R \setminus \bar{\Omega}) \cap \mathrm{BV}(B_R \setminus \bar{\Omega})$, and that if $\mu \in \mathbb{R}$ and we define

$$Q_\mu := \{x \in B_R \setminus \bar{\Omega}: \ \operatorname{div} \xi_{\min}(x) > \mu\},$$

where we can assume that Q_μ has finite perimeter, then

$$\int_{Q_\mu} \operatorname{div} \xi_{\min} \, dx = \mathscr{H}^1(\partial^* Q_\mu \cap \partial\Omega) + c\mathscr{H}^1(\partial^* Q_\mu \cap \partial B_R) - P(Q_\mu, B_R \setminus \bar{\Omega}). \tag{84}$$

We claim that $\operatorname{div} \xi_{\min}$ is constant on $B_R \setminus \bar{\Omega}$, and therefore $\operatorname{div} \xi_{\min} = 0$ on $B_R \setminus \bar{\Omega}$ in view of the choice of c in (80). Suppose by contradiction that $\operatorname{div} \xi_{\min}$ is not identically zero on $B_R \setminus \bar{\Omega}$. By (80) and the Gauss–Green Theorem, it follows that $\{\operatorname{div} \xi_{\min} < 0\}$ cannot be the whole of $B_R \setminus \bar{\Omega}$. It follows that there exists $\lambda > 0$ such that Q_λ is a nonempty set of finite perimeter. Using (84) with $\mu = \lambda$ and (80), the inequality

$$\int_{Q_\lambda} \operatorname{div} \xi_{\min} \, dx > \lambda |Q_\lambda| > 0$$

implies

$$P(Q_\lambda, B_R \setminus \bar{\Omega}) < \mathscr{H}^1(\partial^* Q_\lambda \cap \partial\Omega) - \frac{P(\Omega)}{2\pi R} \mathscr{H}^1(\partial^* Q_\lambda \cap \partial B_R), \tag{85}$$

that is

$$2P(Q_\lambda, B_R \setminus \bar{\Omega}) < P(Q_\lambda, B_R) - \frac{P(\Omega)}{2\pi R} \mathscr{H}^1(\partial^* Q_\lambda \cap \partial B_R). \tag{86}$$

We now split the proof into three cases.

Case 1. Assume $\partial^* Q_\lambda \cap \partial\Omega = \emptyset$. In this case we have $P(Q_\lambda, B_R \setminus \bar{\Omega}) = P(Q_\lambda, B_R)$, which inserted in (86) gives a contradiction.

Case 2. Assume that $\partial^* Q_\lambda \cap \partial B_R = \emptyset$. In this case we have $P(Q_\lambda, B_R \setminus \bar{\Omega}) = P(Q_\lambda, \mathbb{R}^2 \setminus \bar{\Omega})$ and $P(Q_\lambda, B_R) = P(Q_\lambda)$, so that (86) implies

$$2P(Q_\lambda, \mathbb{R}^2 \setminus \bar{\Omega}) < P(Q_\lambda),$$

which contradicts (78) with $D = Q_\lambda$.

Case 3. Assume that $\partial^* Q_\lambda \cap \partial\Omega \neq \emptyset$ and $\partial^* Q_\lambda \cap \partial B_R \neq \emptyset$. By the additivity of the perimeter on connected components, there exists a connected component C of Q_λ such that (85) holds with C in place of Q_λ. On the other hand, using the fact that C is connected, (79), and $|c| < 1$, we get

$$\begin{aligned} P(C, B_R \setminus \bar{\Omega}) &\geqslant 2\,\mathrm{dist}(\partial B_R, \partial\Omega) \geqslant P(\Omega) \\ &> \mathscr{H}^1(\partial^* C \cap \partial\Omega) - \frac{P(\Omega)}{2\pi R} \mathscr{H}^1(\partial^* C \cap \partial B_R), \end{aligned}$$

which contradicts (85).

Our claim is proved, and therefore $\mathrm{div}\, \xi_{\min} = 0$ on $B_R \setminus \bar{\Omega}$. We now extend $\xi_{\min}$ on the whole of $\mathbb{R}^2$ as follows. Define $\xi^+_{\mathbb{R}^2 \setminus \bar{\Omega}}(x) := -\frac{P(\Omega)}{2\pi} \frac{x}{|x|^2}$ if $x \in \mathbb{R}^2 \setminus \bar{B}_R$, and $\xi^+_{\mathbb{R}^2 \setminus \bar{\Omega}}(x) := \xi_{\min}(x)$ if $x \in B_R \setminus \bar{\Omega}$. Finally, define $\xi^+_{\mathbb{R}^2 \setminus \bar{\Omega}}$ inside Ω as follows: first we extend $\xi^+_{\mathbb{R}^2 \setminus \bar{\Omega}}$ in a Lipschitz way, inside Ω, in a suitable open tubular neighborhood of $\partial\Omega$, keeping the constraint $\|\xi\|_\infty = 1$. It is then enough to use a cut-off function to further extend the vector field on the whole of Ω, keeping all required constraints. One can check that $\xi^+_{\mathbb{R}^2 \setminus \bar{\Omega}} \in H^{\mathrm{div}}(\mathbb{R}^2)$, $\|\xi^+_{\mathbb{R}^2 \setminus \bar{\Omega}}\|_\infty \leqslant 1$, and $\mathrm{div}\, \xi^+_{\mathbb{R}^2 \setminus \bar{\Omega}} = 0$ on $\mathbb{R}^2 \setminus \bar{\Omega}$. It follows that $\mathbb{R}^2 \setminus \bar{\Omega}$ is $+$ calibrable. ∎

Remark 10. If the set Ω in Theorem 5 is convex, then (78) is automatically satisfied.

The following theorem generalizes Theorem 4 to nonconnected sets.

THEOREM 6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set of finite perimeter. If $v := \chi_\Omega$ is a solution of* (64), *then Ω has a finite number of connected components $C_1, \ldots, C_m$, and*

(i) *C_i is convex for any $i = 1, \ldots, m$;*

(ii) *∂C_i is of class $C^{1,1}$ for any $i = 1, \ldots, m$;*

(iii) *the following inequalities hold*:

$$\operatorname*{ess\,sup}_{p \in \partial C_i} \kappa_{\partial C_i}(p) \leqslant \frac{P(C_i)}{|C_i|} \qquad \forall i = 1, \ldots, m;$$

(iv) $\frac{P(C_i)}{|C_i|} = \frac{P(C_j)}{|C_j|}$ *for any* $i, j \in \{1, \ldots, m\}$;

(v) *let* $0 \leqslant k \leqslant m$ *and let* $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ *be any* k*-uple of indices*; *if we denote by* $E_{i_1,\ldots,i_k}$ *a solution of the variational problem*

$$\min\left\{P(E)\colon E \text{ of finite perimeter}, \bigcup_{j=1}^{k} C_{i_j} \subseteq E \subseteq \mathbb{R}^2 \setminus \bigcup_{j=k+1}^{m} C_{i_j}\right\}, \tag{87}$$

we have

$$P(E_{i_1,\ldots,i_k}) \geqslant \sum_{j=1}^{k} P(C_{i_j}). \tag{88}$$

Conversely, *assume that* $\Omega \subset \mathbb{R}^2$ *is a bounded open set which is union of a finite number* $C_1, \ldots, C_m$ *of connected components satisfying* (i)–(v). *Then* $v := \chi_\Omega$ *is a solution of* (64).

Proof. Assume that (χ_Ω, ξ) is a solution of (64). By Lemma 3 we have that Ω is $-$ calibrable and $\mathbb{R}^2 \setminus \Omega$ is $+$ calibrable. By Proposition 5 (b) we have that each connected component C of Ω is convex, and by Proposition 6 we have that ∂C is of class $C^{1,1}$. By Remark 6 we have that Ω satisfies (68) so that, by Remark 9, Ω minimizes $\mathscr{G}_{\lambda_\Omega}$ among all finite perimeter subsets of Ω. Thanks to the results in [27], this is equivalent to (69). Therefore, as Ω is bounded, it follows that Ω consists of a finite number of connected components $C_1, \ldots, C_m$. Integrating $-\mathrm{div}\,\xi$ on each C_i we obtain

$$\lambda_\Omega = \lambda_{C_i} = \lambda_{C_j} \qquad \forall i, j \in \{1, \ldots, m\}.$$

It is not difficult to prove that (87) admits a solution. Moreover, this solution is in general not unique; however, since the portions of the boundary of a minimizer which are not contained in $\bigcup_{i=1}^{N} \partial C_i$ are segments, it is possible to prove that the number of different solutions of (87) is finite. Let us now prove (88). Set

$$D := E_{i_1,\ldots,i_k} \setminus \bigcup_{j=1}^{k} C_{i_j} \subseteq \mathbb{R}^2 \setminus \Omega.$$

We have

$$\begin{aligned} 0 = \int_D \mathrm{div}\,\xi \, dx &\geqslant -P(E_{i_1,\ldots,i_k}, \mathbb{R}^2 \setminus \bar{\Omega}) + \mathscr{H}^1(\partial^* D \cap \partial\Omega) \\ &\geqslant -P(E_{i_1,\ldots,i_k}, \mathbb{R}^2 \setminus \bar{\Omega}) + \mathscr{H}^1\left(\partial^* D \cap \left(\bigcup_{j=1}^{k} \partial C_{i_j}\right)\right). \end{aligned}$$

Equivalently,

$$\sum_{j=1}^{k} P(C_{i_j}) \leqslant P(E_{i_1,\dots,i_k}, \mathbb{R}^2\setminus\bar{\Omega}) + \sum_{j=1}^{k} P(C_{i_j}) - \mathscr{H}^1\left(\partial^* D \cap \left(\bigcup_{j=1}^{k} \partial C_{i_j}\right)\right). \tag{89}$$

Since the right-hand side of (89) is less than or equal to $P(E_{i_1,\dots,i_k})$, inequality (88) follows.

Assume now that Ω is a bounded open set which is union of a finite number $C_1,\dots,C_m$ of connected components satisfying (i)–(v). Reasoning as in the proof of (75) it follows that each C_i is $-$ calibrable, so that thanks to (iv) it follows that Ω is $-$ calibrable. To prove that $\mathbb{R}^2\setminus\bar{\Omega}$ is $+$ calibrable, we will show that (78) is valid. Let $D \subset \mathbb{R}^2\setminus\bar{\Omega}$ be a bounded set of finite perimeter. Denote by $C_{i_1},\dots,C_{i_k}$ the connected components of Ω whose boundary intersects $\partial^* D$. Let $E_{i_1,\dots,i_k}$ be a minimizer of problem (87). Using (88) and the minimality of $E_{i_1,\dots,i_k}$ we then have

$$\sum_{j=1}^{k} P(C_{i_j}) \leqslant P(E_{i_1,\dots,i_k}) \leqslant P\left(D \cup \bigcup_{j=1}^{k} C_{i_j}\right). \tag{90}$$

Observe now that

$$\begin{aligned} P\left(D \cup \bigcup_{j=1}^{k} C_{i_j}\right) &= P(D, \mathbb{R}^2\setminus\bar{\Omega}) + \sum_{j=1}^{k} P(C_{i_j}) - \mathscr{H}^1\left(\partial^* D \cap \left(\bigcup_{j=1}^{k} \partial C_{i_j}\right)\right) \\ &= 2P(D, \mathbb{R}^2\setminus\bar{\Omega}) - P(D) + \sum_{j=1}^{k} P(C_{i_j}), \end{aligned}$$

which, inserted in (90), gives (78). According to Lemma 3 we have that $v := \chi_\Omega$ is a solution of (64).

In order to prove Theorem 7 (without the use of the tools introduced in (81) and (82)) we start with the following observation.

Lemma 8. *Let $\alpha_i > 0$ and $B_i \subseteq \mathbb{R}^2$ be bounded measurable sets, for $i = 1,\dots,m$. Let $g := \sum_{i=1}^m \alpha_i \chi_{B_i}$. Then $\|g\|_* \leqslant 1$ if and only if*

$$\sum_{i=1}^{m} \alpha_i |B_i \cap D| \leqslant P(D) \qquad \forall D \subset \mathbb{R}^2, \quad D \text{ bounded of finite perimeter}. \tag{91}$$

Proof. Assume that $\|g\|_* \leqslant 1$. Let $D \subseteq \mathbb{R}^2$ be a bounded set of finite perimeter. Then

$$\sum_{i=1}^{m} \alpha_i |B_i \cap D| = \int_{\mathbb{R}^2} g\chi_D \, dx \leqslant \int_{\mathbb{R}^2} |D\chi_D| = P(D).$$

Conversely, assume that (91) holds. Let $v \in L^2(\mathbb{R}^2) \cap \mathrm{BV}(\mathbb{R}^2)$ be nonnegative. We have

$$\begin{aligned}\int_{\mathbb{R}^2} gv \, dx &= \sum_{i=1}^{m} \alpha_i \int_0^\infty \int_{\mathbb{R}^2} \chi_{B_i}\chi_{\{v \geqslant t\}} \, dx \, dt = \sum_{i=1}^{m} \alpha_i \int_0^\infty |B_i \cap \{v \geqslant t\}| \, dt \\ &\leqslant \int_0^\infty P(\{v \geqslant t\}) \, dt = \int_{\mathbb{R}^2} |Dv|.\end{aligned}$$

Splitting into the positive and negative parts, the above inequality holds for a generic $v \in L^2(\mathbb{R}^2) \cap \mathrm{BV}(\mathbb{R}^2)$. Therefore $\|g\|_* \leqslant 1$. ∎

The following result is essentially a generalization of Theorem 6.

THEOREM 7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded set of finite perimeter and assume that Ω consists of a finite number of connected components $C_1, \dots, C_m$. Let $b_i > 0$ for $i = 1, \dots, m$. The function $u := \sum_{i=1}^{m} b_i \chi_{C_i}$ is a solution of* (4) *if and only if*

(a) $b_i = \frac{P(C_i)}{|C_i|}$ *for all* $i = 1, \dots, m$;
(b) *conditions* (i)–(iii) *and* (v) *of Theorem* 6 *hold.*

Proof. Assume that (u, ξ) is a solution of (4), where $u = \sum_{i=1}^{m} b_i \chi_{C_i}$. The identity $(\xi, Du) = |Du|$ implies that $(\xi, D\chi_{C_i}) = |D\chi_{C_i}|$ as measures in $\mathbb{R}^2$, for all $i = 1, \dots, m$. Using this observation and integrating the equality $-\mathrm{div}\, \xi = u$ in C_i it follows that $b_i = \lambda_{C_i}$. Now, let $D \subseteq \mathbb{R}^2$ be a set of finite perimeter. Multiplying the equation $-\mathrm{div}\, \xi = u$ by χ_D and integrating in $\mathbb{R}^2$ we obtain

$$P(D) \geqslant -\int_{\mathbb{R}^2} \chi_D \mathrm{div}\, \xi \, dx = \sum_{i=1}^{m} b_i |C_i \cap D| \geqslant b_j |C_j \cap D|, \tag{92}$$

i.e., $\lambda_{C_j} \leqslant \frac{P(D)}{|C_j \cap D|}$ for each $j = 1, \dots, m$. As in the proof of Theorem 6, it follows that (i)–(iii) hold. Finally, let us prove that condition (v) holds. If we write (92) for $D = E_{i_1, \dots, i_k}$ we have

$$\sum_{i=1}^{m} \lambda_{C_i} |C_i \cap E_{i_1, \dots, i_k}| \leqslant P(E_{i_1, \dots, i_k}),$$

which gives (6) since $C_{i_j} \cap E_{i_1,\ldots,i_k} = C_{i_j}$ for $j = 1, \ldots, k$, while $C_i \cap E_{i_1,\ldots,i_k} = \emptyset$ for $i \notin \{i_1, \ldots, i_k\}$.

Conversely, assume that conditions (a) and (b) hold. Reasoning as in the proof of (75) it follows that each C_i is $-$ calibrable. We shall prove that $g := \sum_{i=1}^m \lambda_{C_i}\chi_{C_i}$ satisfies $\|g\|_* \leqslant 1$. According to Lemma 8, it will be sufficient to prove that

$$\sum_{i=1}^m \lambda_{C_i}|C_i \cap D| \leqslant P(D) \qquad \forall D \subset \mathbb{R}^2, \quad D \text{ bounded of finite perimeter.} \tag{93}$$

By additivity of the area and the perimeter, it is sufficient to prove (93) when D is also indecomposable. Let $D \subseteq \mathbb{R}^2$ be such a set. Since C_i are $-$ calibrable sets, by Remark 6 (applied with $\Omega := C_i$ and $D := D \cap C_i$), we have that

$$\lambda_{C_i}|C_i \cap D| \leqslant P(C_i \cap D).$$

Then, to prove (93), it will be sufficient to prove that

$$\sum_{i=1}^m P(C_i \cap D) \leqslant P(D) \qquad \forall D \subset \mathbb{R}^2,$$
$$D \text{ bounded indecomposable of finite perimeter.} \tag{94}$$

Denote by $C_{i_1}, \ldots, C_{i_k}$ the connected components of Ω such that $D \cup \bigcup_{j=1}^k C_{i_j}$ is connected. Those components intersect either D or $\partial^* D$. Let $E_{i_1,\ldots,i_k}$ be a minimizer of problem (87). Using (88) and the minimality of $E_{i_1,\ldots,i_k}$ we then have

$$\sum_{j=1}^k P(C_{i_j}) \leqslant P(E_{i_1,\ldots,i_k}) \leqslant P\left(D \cup \bigcup_{j=1}^k C_{i_j}\right). \tag{95}$$

We claim that

$$P\left(D \cup \bigcup_{j=1}^k C_{i_j}\right) \leqslant P(D, \mathbb{R}^2 \setminus \bar{\Omega}) + \sum_{j=1}^k P(C_{i_j}) - \mathscr{H}^1\left(D \cap \left(\bigcup_{j=1}^k \partial C_{i_j}\right)\right). \tag{96}$$

Indeed, since $\partial^*(D \cup X) \subseteq (\partial^* D \setminus X) \cup (\partial X \setminus D)$ where $X := \bigcup_{j=1}^k C_{i_j}$, we have

$$P(D \cup X) \leqslant \mathscr{H}^1(\partial^* D \setminus X) + \mathscr{H}^1(\partial X \setminus D) - \mathscr{H}^1(\partial^* D \cap \partial X)$$

since the term with a minus sign was counted twice by the first two terms at the right-hand side. Thus

$$\begin{aligned}P(D \cup X) &\leqslant \mathcal{H}^1(\partial^* D \setminus \bar{X}) + \mathcal{H}^1(\partial X \setminus D) = P(D, \mathbb{R}^2 \setminus \bar{X}) + P(X) - \mathcal{H}^1(\partial X \cap D)\\ &= P(D, \mathbb{R}^2 \setminus \bar{\Omega}) + P(X) - \mathcal{H}^1(\partial X \cap D)\end{aligned}$$

which proves claim (96).

Inserting (96) into (95), we obtain

$$\mathcal{H}^1\left(D \cap \left(\bigcup_{j=1}^{k} \partial C_{i_j}\right)\right) \leqslant P(D, \mathbb{R}^2 \setminus \bar{\Omega}). \tag{97}$$

On the other hand, since $\partial^*(C_i \cap D) \subseteq (\partial^* D \cap C_i) \cup (\partial C_i \cap D) \cup (\partial^* D \cap \partial C_i)$, we have, using (97),

$$\begin{aligned}\sum_{i=1}^{N} P(C_i \cap D) = \sum_{j=1}^{k} P(C_{i_j} \cap D) &\leqslant P(D, \Omega) + \mathcal{H}^1\left(D \cap \left(\bigcup_{j=1}^{k} \partial C_{i_j}\right)\right)\\ &\quad + \mathcal{H}^1\left(\partial^* D \cap \left(\bigcup_{j=1}^{k} \partial C_{i_j}\right)\right)\\ &\leqslant P(D, \Omega) + P(D, \mathbb{R}^2 \setminus \bar{\Omega}) + \mathcal{H}^1\left(\partial^* D \cap \left(\bigcup_{j=1}^{k} \partial C_{i_j}\right)\right) = P(D).\end{aligned}$$

We have proved that $\|g\|_* \leqslant 1$. According to Lemma 1 there is a vector field $\xi \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ with $\|\xi\|_\infty \leqslant 1$ such that $-\operatorname{div} \xi = u$. Multiplying this equation by u and integrating in $\mathbb{R}^2$ we obtain

$$\int_{\mathbb{R}^2} (\xi, Du) = \int_{\mathbb{R}^2} u^2\, dx = \sum_{i=1}^{m} \frac{P(C_i)^2}{|C_i|} = \int_{\mathbb{R}^2} |Du|.$$

Therefore, u is a solution of (4). ∎

10. EXPLICIT SOLUTIONS FOR THE DENOISING PROBLEM

PROPOSITION 7. *Let $\lambda > 0$, $b \in \mathbb{R}$ and $a := \operatorname{sign}(b)(|b| - \lambda)^+$. If $\bar{u} \in \mathrm{BV}(\mathbb{R}^2)$ is a solution of* (4) *then the function $a\bar{u}$ is the solution of the variational problem* (7) *with $f := b\bar{u}$. Conversely, if $a\bar{u}$ is the solution of* (7) *with $f = b\bar{u}$ and $b - a = \pm\lambda$, then $\bar{u} \in \mathrm{BV}(\mathbb{R}^2)$ is a solution of* (4).

In particular, if Ω satisfies the conditions listed in Theorem 6, *then $a\lambda_\Omega\chi_\Omega$ is a solution of* (7) *with $f = b\lambda_\Omega\chi_\Omega$. The converse statement holds if $b - a = \pm\lambda$.*

Proof. Recall (see Lemma 1) that a function $u \in \mathrm{BV}(\mathbb{R}^2)$ is the solution of (7) if and only if u is the solution of

$$u - \lambda \operatorname{div}\left(\frac{Du}{|Du|}\right) = f. \tag{98}$$

Let $f := b\bar{u}$ where $\bar{u}$ satisfies (4). Without loss of generality, we may assume that $b \geqslant 0$ (the case $b < 0$ can be obtained by changing $b \to -b$ and $u \to -u$). Suppose first that $b > \lambda$, so that $a = b - \lambda$. Since

$$-\lambda \operatorname{div}\left(\frac{D\bar{u}}{|D\bar{u}|}\right) = \lambda\bar{u} = (b-a)\bar{u},$$

it follows that $u := a\bar{u}$ satisfies (98). Now, assume that $0 \leqslant b \leqslant \lambda$, so that $a = 0$. Let $\xi \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ be such that $\|\xi\|_\infty \leqslant 1$ and $-\operatorname{div}\xi = \bar{u}$. Obviously, if $z := \frac{b}{\lambda}\xi$, then $\|z\|_\infty \leqslant 1$, and $-\operatorname{div} z = -\frac{b}{\lambda}\operatorname{div}\xi = \frac{b}{\lambda}\bar{u}$, that is $-\lambda \operatorname{div} z = b\bar{u} = f$. Since $\int_{\mathbb{R}^N}(z, D0) = 0 = \int_{\mathbb{R}^N}|D0|$, it follows that $u = 0$ solves (98). The converse statement follows by substituting $f = b\bar{u}$ and $u = a\bar{u}$ into (98).

The last assertion follows from Theorem 6 and the first part of the proof. ∎

Let us prove an extension of the above result.

PROPOSITION 8. *Let Ω be a bounded set of finite perimeter which consists of a finite number $C_1, \ldots, C_m$ of connected components. Let $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$. Assume that the function $\bar{u} := \sum_{i=1}^m \lambda_{C_i}\chi_{C_i}$ solves* (4). *Let $\lambda > 0$ and $a_i := \operatorname{sign}(b_i)(|b_i| - \lambda)^+$. Then the function $u := \sum_{i=1}^m a_i\lambda_{C_i}\chi_{C_i}$ is the solution of the variational problem* (7) *with $f = \sum_{i=1}^m b_i\lambda_{C_i}\chi_{C_i}$. The converse statement holds if a_i, b_i are such that $b_i - a_i = \lambda$, or $b_i - a_i = -\lambda$, for all $i = 1, \ldots, m$.*

Proof. As in the proof of Proposition 7, we have to prove that u is the solution of (98). We observe that this is obviously true if $b_i \geqslant \lambda$, or $b_i \leqslant -\lambda$, for all $i = 1, \ldots, m$. In the general case, let $I_\lambda := \{i \in \{1, \ldots, m\}\colon |b_i| \geqslant \lambda\}$, $J_\lambda := \{i \in \{1, \ldots, m\}\colon |b_i| < \lambda\}$. Since, in this case,

$$f - u = \lambda \sum_{i \in I_\lambda} \operatorname{sign}(b_i)\lambda_{C_i}\chi_{C_i} + \sum_{i \in J_\lambda} b_i\lambda_{C_i}\chi_{C_i},$$

to prove that u is a solution of (98) we have to construct a vector field $\xi \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ with $\|\xi\|_\infty \leqslant 1$, such that

$$-\operatorname{div}\xi = \sum_{i \in I_\lambda} \operatorname{sign}(b_i)\lambda_{C_i}\chi_{C_i} + \sum_{i \in J_\lambda} \frac{b_i}{\lambda}\lambda_{C_i}\chi_{C_i} \tag{99}$$

and $(\xi, Du) = |Du|$. Let $F \in L^2(\mathbb{R}^2)$ denote the right-hand side of (99), and let $F^+ = \sup(F, 0)$, $F^- = \sup(-F, 0)$. Let $(\bar{u}, \xi_{\bar{u}})$ be a solution of (4). Let $D \subseteq \mathbb{R}^2$ be a set of finite perimeter. Multiplying the equation $-\mathrm{div}\, \xi_{\bar{u}} = \bar{u}$ by χ_D and integrating in $\mathbb{R}^2$ we have that

$$P(D) \geqslant -\int_{\mathbb{R}^2} \mathrm{div}\, \xi_{\bar{u}} \chi_D \, dx = \sum_{i=1}^{m} \lambda_{C_i} \int_{\mathbb{R}^2} \chi_{C_i} \chi_D \, dx = \sum_{i=1}^{m} \lambda_{C_i} |C_i \cap D|. \quad (100)$$

This inequality implies that $\|F\|_* \leqslant 1$. Indeed, let $v \in \mathrm{BV}(\mathbb{R}^2)$. Since

$$\int_{\mathbb{R}^2} F(x)v(x)dx \leqslant \int_{\mathbb{R}^2} (F^+ v^+ + F^- v^-)dx$$

and $\int_{\mathbb{R}^2} |Dv| = \int_{\mathbb{R}^2} |Dv^+| + \int_{\mathbb{R}^2} |Dv^-|$, the inequality $\int_{\mathbb{R}^2} F(x)v(x)\, dx \leqslant \int_{\mathbb{R}^2} |Dv|$ follows if we prove that

$$\int_{\mathbb{R}^2} F^+ v^+ \, dx \leqslant \int_{\mathbb{R}^2} |Dv^+| \quad \text{and} \quad \int_{\mathbb{R}^2} F^- v^- \, dx \leqslant \int_{\mathbb{R}^2} |Dv^-|.$$

Thus, without loss of generality, we may assume that $F \geqslant 0$ and $v \in \mathrm{BV}(\mathbb{R}^2)$, $v \geqslant 0$. Then, using that $\frac{b_i}{\lambda} \leqslant 1$ for any $i \in J_\lambda$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^2} F(x)v(x)\, dx &= \int_0^\infty \int_{\mathbb{R}^2} F \chi_{\{v \geqslant t\}} \, dx\, dt \\
&= \sum_{i \in I_\lambda} \lambda_{C_i} \int_0^\infty \int_{\mathbb{R}^2} \chi_{C_i} \chi_{\{v \geqslant t\}} \, dx\, dt + \sum_{i \in J_\lambda} \frac{b_i}{\lambda} \lambda_{C_i} \int_0^\infty \int_{\mathbb{R}^2} \chi_{C_i} \chi_{\{v \geqslant t\}} \, dx\, dt \\
&\leqslant \sum_{i=1}^{m} \lambda_{C_i} \int_0^\infty |C_i \cap \{v \geqslant t\}| \, dx\, dt \leqslant \int_0^\infty P([v \geqslant t])\, dt = \int_{\mathbb{R}^2} |Dv|.
\end{aligned}$$

Therefore $\|F\|_* \leqslant 1$. By Lemma 1, there is a vector field $\xi \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ such that $\|\xi\|_\infty \leqslant 1$, satisfying (99). Since $a_i = 0$ for all $i \in J_\lambda$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^2} |Du| &= \sum_{i \in I_\lambda} |a_i| \lambda_{C_i} P(C_i) = \sum_{i \in I_\lambda} a_i \lambda_{C_i} \int_{\mathbb{R}^2} (-\mathrm{div}\, \xi) \chi_{C_i} \, dx \\
&= \sum_{i=1}^{m} a_i \lambda_{C_i} \int_{\mathbb{R}^2} (\xi, D\chi_{C_i}) = \int_{\mathbb{R}^2} (\xi, Du),
\end{aligned}$$

which, in turn implies that $(\xi, Du) = |Du|$, since $\|\xi\|_\infty\| \leqslant 1$.

The converse statement is obvious. ∎

Proposition 8 proves that a_i is a soft thresholding of b_i with threshold λ. This is in coincidence with the soft thresholding rule used in the wavelet shrinkage method for denoising [22, 23, 24, 30, 37]. As proved by Meyer [30], a soft thresholding applied to the wavelet coefficients of the function $f \in L^2(\mathbb{R}^2)$ gives a quasi-optimal solution of the denoising problem (7). Let us also mention that it has been proved recently that the wavelet coefficients of a BV function are somewhere between ℓ^1 and weak ℓ^1 [19, 20, 30, 32].

Finally, that a solution of (7) when Ω is a ball was given by the above formula was already observed by Meyer [30] and Strong–Chan [34].

11. SOME EXAMPLES

In order to clarify the conditions given in Sections 8 and 9, we shall discuss some explicit examples.

Example 1. Let $\Omega \subset \mathbb{R}^2$ be the set of Fig. 1. It is easy to check that Ω satisfies the assumptions of Theorem 4, since Ω is a convex set with $C^{1,1}$ boundary and there holds

$$\operatorname*{ess\,sup}_{p\in\partial\Omega} \kappa_{\partial\Omega}(p) = \frac{1}{r} < \frac{2\pi r + 2L}{\pi r^2 + 2rL} = \frac{P(\Omega)}{|\Omega|}. \tag{101}$$

Moreover, since the inequality in (101) is always strict, the solution of (1) starting from $\chi_{\Omega'}$ remains a characteristic function for any convex set Ω' of class $C^{1,1}$ close enough to Ω in the $C^{1,1}$-norm.

Example 2. Let $\Omega \subset \mathbb{R}^2$ be the union of two disjoint balls of radius r, whose centers are at distance L (see Fig. 2). Then condition (88) of Theorem 6 reads as

$$L \geqslant \pi r.$$

Under this condition the solution of (1) and (2) with $u_0 = \chi_\Omega$ remains a characteristic function.

Example 3. Consider now three disjoint balls of radius r, whose centers are on the vertices of an equilateral triangle with edges of length 1 (see Fig. 3). In this case, condition (88) reads as

$$r \leqslant \frac{3}{4\pi}.$$

Notice that this condition is more restrictive than the condition holding for two balls, which has been discussed in Example 1 and gives $r \leqslant \frac{1}{\pi}$. This implies that it is not enough to consider only pairs of sets in condition (v) of Theorem 6.

Example 4. We give now an example of an explicit solution, which is also a solution of (1) which is not among the solutions considered in Sections 8 and 9. Let $\Omega := B_R(0)\setminus\overline{B_r(0)}$ be the set of Fig. 4. In this case Ω does not satisfy assumption (i) of Theorem 4, i.e., Ω is not convex. However, is it possible to compute explicitly the solution of (1) and (2) with $u_0 = \chi_\Omega$. Indeed, let $\xi : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined as

$$\xi(x) := \begin{cases} \frac{x}{r} & \text{for } x \in B_r(0), \\ \left(\frac{Rr}{|x|^2} - 1\right)\frac{x}{R-r} & \text{for } x \in B_R(0)\setminus\overline{B_r(0)}, \\ -\frac{R}{|x|^2}x & \text{for } x \in \mathbb{R}^2\setminus\overline{B_R(0)}. \end{cases}$$

Then $\|\xi\|_\infty \leqslant 1$, $\operatorname{div}\xi = \frac{2}{r}$ on $B_r(0)$, $\operatorname{div}\xi = -\frac{2}{R-r}$ on $B_R(0)\setminus\overline{B_r(0)}$, $\operatorname{div}\xi = 0$ on $\mathbb{R}^2\setminus\overline{B_R(0)}$, and $\xi\cdot\nu^{B_r(0)} = 1$ on $\partial B_r(0)$, $\xi\cdot\nu^{B_R(0)} = -1$ on $\partial B_R(0)$. Therefore, one can check that the solution u of (1) and (2) with $u_0 = \chi_\Omega$ is given by

$$u(t,x) = (1-\lambda_\Omega t)\chi_\Omega(x) + \frac{2t}{r}\chi_{B_r(0)}(x), \qquad t \in \left[0, \frac{r(R-r)}{2R}\right], \quad x \in \mathbb{R}^2.$$

For $t > \frac{r(R-r)}{2R}$ the solution u is equal to the solution starting from $(1-\frac{r}{R})\chi_{B_R(0)}$ (at time $\frac{r(R-r)}{2R}$) and it is one of the solutions described in Sections 8 and 9.

Example 5. Let $0 = R_0 < R_1 < \cdots < R_p < R_{p+1} = +\infty$, so that $B_{R_0}(0) = \emptyset$, $B_{R_{p+1}}(0) = \mathbb{R}^2$. Set for simplicity $B_i := B_{R_i}(0)$, for $i = 0,\ldots,p+1$. Let $\Omega_i := B_i\setminus\bar{B}_{i-1}$, $i = 1,\ldots,p+1$. Let $a_1,\ldots,a_{p+1}$ be real numbers such that $a_i \neq a_{i-1}$, $a_i \neq a_{i+1}$, $i = 2,\ldots,p$, and $a_{p+1} = 0$. Let $\bar{u} := \sum_{i=1}^p a_i\chi_{\Omega_i}$. We claim that choosing a_i appropriately we have that u is a solution of (4). To be more precise, we say that we have specified a qualitative ordering of $a_1,\ldots,a_{p+1}$ if we have said if a_1 is above a_2 (i.e., $a_1 > a_2$) or below a_2 (i.e., $a_1 < a_2$), a_2 is above or below $a_3,\ldots,a_p$ is above or below a_{p+1}. Then, for each qualitative ordering of $a_1,\ldots,a_{p+1}$, the values of $a_1,\ldots,a_{p+1}$ can be uniquely specified so that u is a solution of (4). This will be a consequence of the following observations.

If $(\bar{u}, z)$, with $\bar{u} = \sum_{i=1}^p a_i\chi_{\Omega_i}$, is a solution of (4), then integrating $\operatorname{div} z$ in B_i we get

$$\int_{\partial B_i} z\cdot\nu^{B_i}\, d\mathcal{H}^1 = \varepsilon_i P(B_i), \tag{102}$$

where $\varepsilon_i := \operatorname{sign}(a_{i+1} - a_i)$. Now, integrating (4) in Ω_i and using (102) we obtain

$$a_i = \frac{\varepsilon_{i-1} P(B_{i-1}) - \varepsilon_i P(B_i)}{|B_i| - |B_{i-1}|}, \tag{103}$$

where $P(B_0) = 0$ and $|B_0| = 0$.

If $B_R := B_R(0)$, we recall that the vector fields $\xi(x) := \frac{x}{R}$ and $z(x) := R\frac{x}{|x|^2}$ satisfy

$$-\operatorname{div}\xi = \frac{P(B_R)}{|B_R|} \qquad \text{in } B_R, \qquad \xi|_{\partial B_R} = \frac{x}{|x|},$$

respectively,

$$-\operatorname{div} z = 0 \qquad \text{in } \mathbb{R}^2 \setminus \bar{B}_R, \qquad z|_{\partial B_R} = \frac{x}{|x|}.$$

The following lemma follows by a simple computation and we shall omit its proof.

LEMMA 9. *Let $0 < r < R$. The vector field $\xi^{-,-}(x) = -(1 + \frac{rR}{|x|^2})\frac{x}{R+r}$ satisfies*

$$-\operatorname{div}\xi^{-,-} = \frac{P(B_R) - P(B_r)}{|B_R| - |B_r|} \quad \text{in } B_R \setminus \bar{B}_r, \qquad \xi|_{\partial B_R} = -\frac{x}{|x|}, \qquad \xi|_{\partial B_r} = -\frac{x}{|x|}.$$

The vector field $\xi^{-,+}(x) := (\frac{rR}{|x|^2} - 1)\frac{x}{R-r}$ satisfies

$$-\operatorname{div}\xi^{-,+} = \frac{P(B_R) + P(B_r)}{|B_R| - |B_r|} \quad \text{in } B_R \setminus \bar{B}_r, \qquad \xi|_{\partial B_R} = -\frac{x}{|x|}, \qquad \xi|_{\partial B_r} = \frac{x}{|x|}.$$

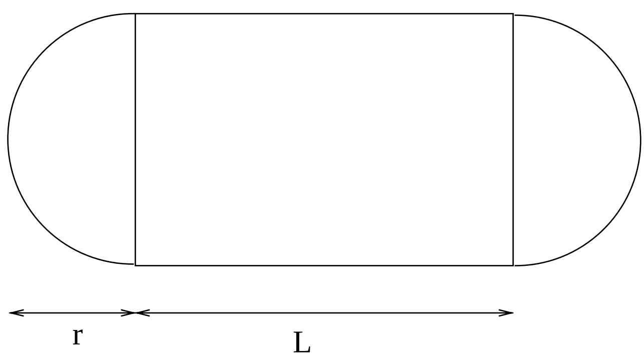

FIG. 1. A bean-shaped set as initial datum for the solution.

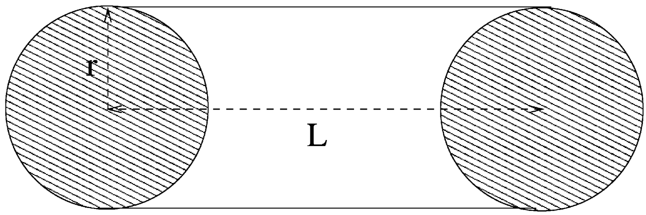

FIG. 2. Two balls as initial datum for the solution.

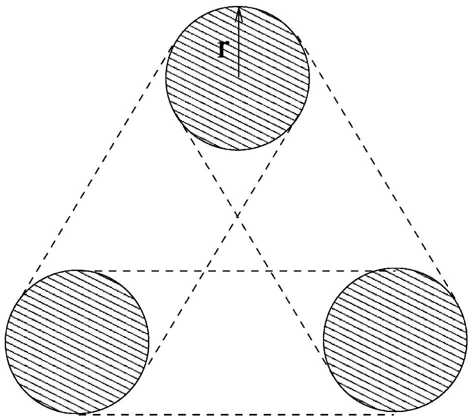

FIG. 3. Three balls as initial datum for the solution.

The vector field $\xi^{+,-}(x) := (1 - \frac{rR}{|x|^2})\frac{x}{R-r}$ *satisfies*

$$-\mathrm{div}\,\xi^{+,-} = -\frac{P(B_R) + P(B_r)}{|B_R| - |B_r|} \quad in\ B_R \backslash \bar{B}_r, \qquad \xi|_{\partial B_R} = \frac{x}{|x|}, \qquad \xi|_{\partial B_r} = -\frac{x}{|x|}.$$

The vector field $\xi^{+,+}(x) = (1 + \frac{rR}{|x|^2})\frac{x}{R+r}$ *satisfies*

$$-\mathrm{div}\,\xi^{+,+} = -\frac{P(B_R) - P(B_r)}{|B_R| - |B_r|} \quad in\ B_R \backslash \bar{B}_r, \qquad \xi|_{\partial B_R} = \frac{x}{|x|}, \qquad \xi|_{\partial B_r} = \frac{x}{|x|}.$$

In all cases $\|\xi^{\pm,\pm}\|_\infty \leqslant 1$.

Finally, let us check that given a qualitative ordering of $a_1, \ldots, a_{p+1}$ there is a corresponding solution of (4) of the form $\bar{u} = \sum_{i=1}^{p} a_i \chi_{\Omega_i}$. First, we observe that once we have specified ε_1, the value of a_1 is given by $a_1 = -\varepsilon_1 \frac{P(B_1)}{|B_1|}$. Thus, it will be sufficient to check that given three consecutive values a_{i-1}, a_i, a_{i+1} with their qualitative ordering, we can uniquely determine the value of a_i. For simplicity, let us denote these values as a_1,

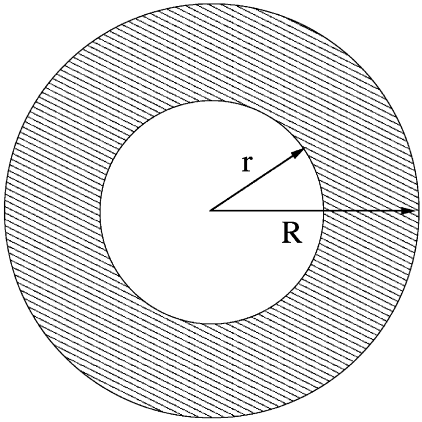

FIG. 4. An explicit solution starting from a ring.

a_2, a_3. Let us prove the compatibility of the values of a_1, a_2, a_3 given by (103) with its qualitative ordering, if this is specified in advance. There are four cases to be considered: (i) $a_3 < a_2$, $a_1 < a_2$, (ii) $a_3 < a_2$, $a_1 > a_2$, (iii) $a_3 > a_2$, $a_1 > a_2$, (iv) $a_3 > a_2$, $a_1 < a_2$.

Assume that we are in case (i). Then $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. Then, by Lemma 9, we have

$$a_0 = \frac{\varepsilon_0 P(B_0) - P(B_1)}{|B_1| - |B_0|}, \qquad a_2 = \frac{P(B_2) + P(B_1)}{|B_2| - |B_1|}, \qquad a_3 = \frac{-P(B_2) - \varepsilon_3 P(B_3)}{|B_3| - |B_2|}.$$

Independently of the values of $\varepsilon_0, \varepsilon_3 \in \{+1, -1\}$ we have

$$a_1 \leqslant \frac{P(B_0) - P(B_1)}{|B_1| - |B_0|} < a_2, \qquad a_3 \leqslant \frac{-P(B_2) + P(B_3)}{|B_3| - |B_2|} < a_2.$$

Thus, the value of a_2 is consistent with the qualitative ordering specified in advance. The other three cases can be checked in a similar way. Thus, having specified the qualitative ordering of $a_1, \ldots, a_{p+1}$, the values of ε_i are given, and formula (103) gives the corresponding value of a_i. We have checked the consistency of this choice. In that case, $\bar{u} = \sum_{i=1}^{p} a_i \chi_{\Omega_i}$ is a solution of (4) and, by Proposition 7, $u = a\bar{u}$ is a solution of (7) with $f = b\bar{u}$, and $a = \mathrm{sign}(b)(|b| - \lambda)^+$. The same result, with a similar proof, can be proved in $\mathbb{R}^N$. This result has already been observed by Strong–Chan [34].

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REFERENCES

1. L. Ambrosio, V. Caselles, S. Masnou, and J.-M. Morel, Connected components of sets of finite perimeter and applications to image processing, *J. European Math. Soc.* **3** (2001), 39–92.
2. L. Ambrosio, N. Fusco, and D. Pallara, "Functions of bounded variation and free discontinuity problems," Oxford Mathematical Monographs, Clarendon Press, Oxford University Press, New York, 2000.
3. F. Andreu, C. Ballester, V. Caselles, and J. M. Mazón, Minimizing total variational flow, *Differential Integral Equations* **4** (2001), 321–360.
4. F. Andreu, C. Ballester, V. Caselles, and J. M. Mazón, The Dirichlet problem for the total variational flow, *J. Funct. Anal.* **180** (2001), 347–403.
5. F. Andreu, V. Caselles, J. I. Diaz, and J. M. Mazón, Qualitative properties of the total variation flow, preprint, 2000.
6. F. Andreu-Vaillo, V. Caselles, and J. M. Mazón, Existence and uniqueness of solutions for a parabolic quasilinear problem for linear growth functionals with L^1 data, *Math. Ann.* **322** (2002), 139–206.
7. F. Andreu, J. M. Mazón, S. Segura, and J. Toledo, Existence and uniqueness for a degenerate parabolic equation with L^1-data, *Trans. Amer. Math. Soc.* **315** (1999), 285–306.
8. G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, *Ann. Mat. Pura Appl. (4)* **135** (1983), 293–318.
9. G. Bellettini, M. Novaga, and M. Paolini, On a crystalline variational problem, part I: first variation and global L^∞-regularity, *Arch. Rational Mech. Anal.* **157** (2001), 165–191.
10. G. Bellettini, M. Novaga, and M. Paolini, On a crystalline variational problem, part II: BV-regularity and structure of minimizers on facets, *Arch. Rational Mech. Anal.* **157** (2001), 193–217.
11. G. Bellettini, M. Novaga, and M. Paolini, Characterization of facet breaking for nonsmooth mean curvature flow in the convex case, *Interfaces Free Boun.* **3** (2001), 415–446.
12. G. Bellettini and M. Paolini, Anisotropic motion by mean curvature in the context of Finsler geometry, *Hokkaido Math. J.* **25** (1996), 537–566.
13. Ph. Benilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, and J. L Vazquez, An L^1-theory of existence and uniqueness of solution of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **XXII** (1995), 241–273.
14. Ph. Benilan and M. G. Crandall, Regularizing effects in homogeneous equations, *in* "Contributions to Analysis and Geometry," (D. N. Clark et al., Eds.), pp. 23–39, John Hopkins Univ. Press, Baltimore, MD, 1981.
15. Ph. Benilan and M. G. Crandall, Completely accretive operators, *in* "Semigroups Theory and Evolution Equations," (Ph. Clement et al., Eds.), pp. 41–76, Marcel Dekker, New York, 1991.
16. H. Brezis, "Operateurs Maximaux Monotones," North-Holland, Amsterdam, 1973.
17. J. Carrillo, On the uniqueness of solution of the evolution dam problem, *Nonlinear Anal.* **22** (1994), 573–607.

18. A. Chambolle and P. L. Lions, Image recovery via total variation minimization and related problems, *Numer. Math.* **76** (1997), 167–188.
19. A. Cohen, W. Dahmen, I. Daubechies, and R. De Vore, Harmonic analysis and the space BV(R^2), preprint, 2000.
20. A. Cohen, R. De Vore, P. Petrushev, and H. Xu, Nonlinear approximation and the space BV, *Amer. J. Math.* **121** (1999), 587–628.
21. M. G. Crandall and T. M. Liggett, Generation of semigroups of nonlinear transformations on general banach spaces, *Amer. J. Math.* **93** (1971), 265–298.
22. D. Donoho, Denoising via soft-thresholding, *IEEE Trans. Inform. Theory* **41** (1995), 613–627.
23. D. Donoho, Nonlinear solution of linear inverse problems by wavelet–vaguelette decomposition, *Appl. Comput. Harmonic Anal.* **2** (1995), 101–126.
24. D. Donoho, I. Johnstone, G. Kerkyacharian, and D. Picard, Wavelet shrinkage: asymptopia ? *J. Roy. Statist. Soc. B* **57** (1995), 301–369.
25. S. Durand, F. Malgouyres, and B. Rougé, Image de-blurring, spectrum interpolation and application to satellite imaging, *Control*, *Optim. Calc. Var.* **5** (2000), 445–475.
26. L. C. Evans and R. F. Gariepy, "Measure Theory and Fine Properties of Functions," Studies in Advanced Mathematics, CRC Press, Ann Arbor, 1992.
27. E. Giusti, On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions, *Invent. Math.* **46** (1978), 111–137.
28. L. Hörmander, "Notions of Convexity," Birkhäuser, Boston, 1994.
29. S. N. Kruzhkov, First order quasilinear equations in several independent variables, *Math. USSR-Sb.* **10** (1970), 217–243.
30. Y. Meyer, Oscillating patterns in image processing and in some nonlinear evolution equations, preprint, 2001.
31. M. Nikolova, Local strong homogeneity of a regularized estimator, *SIAM J. Appl. Math.* **61** (2000), 633–658.
32. F. Oru, "Le Role des Oscillations Dans Quelques Problêms D'analyse Non-Linéaire," Ph.D. thesis, Thése CMLA, ENS-Cachan, France, 1998.
33. L. Rudin, S. Osher, and E. Fatemi, Nonlinear total variation based noise removal algorithms, *Physica D* **60** (1992), 259–268.
34. D. M. Strong and T. F. Chan, "Exact Solutions to Total Variation Regularization problems," CAM Report, 1996.
35. G. H. Golub, T. F. Chan, and P. Mulet, A nonlinear primal-dual method for total variation based image restoration, *SIAM J. Sci. Comput.* **20** (1999), 1964–1977.
36. C. R. Vogel and M. E. Oman, Fast total variation based image reconstruction, *in* "Proceedings of the 1995 ASME Design Engineering Conferences, Boston, MA," Vol. 3, pp. 1009–1015, 1995.
37. R. A. De Vore and B. J. Lucier, Fast wavelet techniques for near optimal image processing, *in* "IEEE Military Communications Conference, Record," San Diego, October 11–14, IEEE, Piscataway, NJ, pp. 1129–1135, 1992.
38. W. P. Ziemer, "Weakly Differentiable Functions," Springer-Verlag, Berlin, 1989.